Fibrations, Logical Predicates and Indeterminates

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Abstract

Within the framework of categorical logic or categorical type theory, predicate logics and type theories are understood as fibrations with structure. Fibrations, or fibred categories, provide an abstract account of the notions of indexing and substitution. These notions are central to the interpretations of predicate logics and type theories with dependent types or polymorphism. In these systems, predicates/dependent types are indexed by the contexts which declare the types of their free variables, and there is an operation of substitution of terms for free variables.

With this setting, it is natural to give a category-theoretic account of certain logical issues in terms of fibrations. In this thesis we explore logical predicates for simply typed theories, induction principles for inductive data types, and indeterminate elements for fibrations in relation to polymorphic λ-calculi.

The notion of logical predicate is a useful tool in the study of type theories like simply typed λ-calculus. For a categorical account of this concept, we are led to study certain structure of fibred categories. In particular, the kind of structure involved in the interpretation of simply typed λ-calculus,

namely cartesian closure, is expressed in terms of adjunctions. Hence we are led to consider adjunctions between fibred categories. We give a characterisation of these adjunctions which allows us to provide structure, given by adjunctions, to a fibred category in terms of appropriate structure on its base and its fibres.

By expressing the abovementioned categorical structure logically, in the internal language of a fibration, we can give an account of logical predicates for a cartesian closed category. By recourse to the internal language, we regard a fibred category as a category of predicates. With the same method, we provide a categorical interpretation of the induction principle for inductive data types, given by initial algebras for endofunctors on a distributive category.

We also consider the problem of adjoining indeterminate elements to fibrations. The category-theoretic concept of indeterminate or generic element captures the notion of parameter. Lambek applied this concept to characterise a functional completeness property of simply typed λ-calculus or, equivalently, of cartesian closed categories. He showed that cartesian categories with indeterminate elements correspond to Kleisli categories for suitable comonads. Here we generalise this result to account for indeterminates for cartesian objects in a 2-category with suitable structure. To specialise this 2-categorical formulation of objects with indeterminates via Kleisli objects to the 2-category $\mathcal{F}ib$ of fibrations over arbitrary bases, we are led to show the existence of Kleisli objects for fibred comonads.These results provide us with the appropriate machinery to study functional completeness for polymorphic λ-calculi by means of fibrations with indeterminates. These are also applied to give a semantics to ML module features: signatures, structures and functors.

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Most diagrams in this thesis were drawn with Paul Taylor's macros.

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Declaration

This thesis was composed by myself and the work reported in it is my own except where otherwise stated.

Contents

Introduction

Categorical logic or categorical type theory, as presented in [Jac91a] for instance, is the application of category theory to the understanding (semantics, relative interpretations, independence results, etc.) of logics and type theory. Type theory subsumes logic via the propositions-as-types paradigm, also known as the Curry-Howard isomorphism [How80], which identifies a proposition with the type of its proofs. This is the point of view adopted in categorical logic; if we only care about entailment between propositions (a proof-irrelevant approach) propositions become with at most one element.

Category theory is convenient to study non-conventional logics like several kinds of lambda calculi. This has been a major application of category theory in computer science. Following the categorical logic approach, category theory gives abstract denotational semantics for programming languages and their associated logics [BW90, AL91, KN93]. The paradigmatic example of such application is the semantics of the simply typed λ-calculus, which can be regarded as a primitive typed functional programming language, as explained for instance in [Mit90]. The idea is that data types of the programming language correspond to types of λ-calculus, while programs correspond to terms. The simply typed λ-calculus can be described in terms of cartesian closed categories, as in [LS86]. The interpretation regards types as objects and terms, or 'programs', as morphisms. The various type and term constructors are described by the product and exponential adjunctions. Such an interpretation yields insight into the essential features of the language, by providing an abstract, syntax-free presentation of these.

In predicate logics and type theories, where predicates, types and terms may involve free variables, contexts are used to provide types for such variables. Hence, contexts have a structural role: every entity with free variables is given relative to a typing context. We view such entities as indexed by the contexts. The operation of substitution of a term for a variable in a predicate/type/term is characteristic of these systems. The categorical study of such systems must therefore account for the notions of indexing and substitution. The appropriate categorical structure to understand these concepts is that of *fibration* or fibred category [Gro71]. It consists of a functor $p : \mathbb{E} \to \mathbb{B}$ satisfying a certain 'cartesian lifting' property. Within the framework of fibrations, the usual logical connectives and quantifiers are modelled by fibred adjunctions, a notion which plays a central role in this thesis.

In this setting, is natural to interpret logical issues categorically by looking at properties and constructions in the 2-category $\mathcal{F}ib$, the 'universe of fibrations'. In this thesis we consider logical predicates for simply typed λ-calculus, the induction principle for inductive data types, and indeterminates for fibrations. We comment on these topics below.

The notion of logical predicate, as in [Mit90], is an important tool in the study of metatheoretic properties of type theories like the simply typed λ-calculus, *e.g.* strong normalisation. It has several applications in programming language semantics. [Abr90] uses logical relations to relate concrete and abstract interpretations of a simple programming language and thus establish correctness of certain analyses of program properties, like strictness and termination analysis. [OT93] use a relational semantics based on logical relations to obtain models for local variables which validate desirable operational equivalences between programs. [Rey83] proposed the use of logical relations to characterise parametric polymorphism, by requiring the identity relation to be logical, in a sense adequate for system F. A logical predicate over a model of the simply typed λ-calculus consists of a collection of predicates, one for each type in the system, satisfying certain conditions. Logical predicates could be used to provide a 'relational semantics' (types-as-relations) for λ-calculi, as explained in [MS92].

The original definition of logical predicates is couched in set-theoretic terms, for Henkin-type models, as given in [Mit90]. It is convenient for a better understanding of the meaning of logical predicates, in particular with a view to their generalisation to other systems, to give an abstract account of them. An intended categorical account of logical predicates appears in [MR91]. The authors introduce a 'category of relations' $\mathcal{R}el$ over a base category $\mathbb{B}$, with a forgetful functor $U : \mathcal{R}el \to \mathbb{B}$, intended as a direct generalisation of the category $Sub(\mathcal{S}et)$ consisting of sets equipped with a distinguished subset/predicate and functions which respect such subsets, with the forgetful functor $\iota : Sub(\mathcal{S}et) \to \mathcal{S}et$. Their category of relations $\mathcal{R}el$ is based on the notion of subobject. The point is that, under certain assumptions, the category of relations has the appropriate structure to interpret the type theory under consideration, *e.g.* cartesian closure for simply typed λ-calculus. A similar approach is taken in [MS92].

Here we take a more abstract approach, based on the observation that the abovementioned categories of relations are *fibred* over their base categories: $U : \mathcal{R}el \to \mathbb{B}$ is a fibration. The relevant structure, *e.g.* cartesian closure of $\mathcal{R}el$, required for the interpretation of logical predicates in such categories arises precisely because they are fibred.

This latter aspect of 'categories of relations' is therefore central to our approach: not only does it allow us to give a precise connection between logical predicates and categorical structure, via the internal language of the fibration, but it also allows us to formulate suitably abstract results which show the conceptual unity of the various constructions involved. Specifically, the cartesian closed structure of a category is given in terms of adjunctions. This leads us to consider adjunctions between fibrations, over possibly different bases. A main technical result characterises these adjunctions in terms of adjunctions between the base categories and vertical fibred adjunctions, involving the change-of-base construction (Theorem 3.2.3). This gives as a simple consequence the 'lifting' of cartesian closed structure from $\mathbb{B}$ to $\mathcal{R}el$, Corollary 3.3.11. This lifting of cartesian closure could be proved directly, but the existence of the abovementioned characterisation puts the result in its appropriate context.

Exploiting the above mentioned relationship between logical concepts and categorical structure in a fibred category, we give a categorical formulation of the induction principle for inductive data types. We adopt the simple approach in [Jac93, CS91] and present inductively defined data types as initial algebras for an endofunctor in a distributive category. The formulation is in the same spirit as Reynold's identity extension lemma [MR91]. It asserts the validity of the induction principle in a fibration by requiring a functor to preserve some structure, namely initial algebras.

We then consider the problem of adjoining indeterminate elements to fibrations. The aim is to generalise to the fibred setting the following situation: given a category $\mathbb{C}$, with a terminal object 1, and an object I, we can construct the so-called polynomial category $\mathbb{C}[x : I]$, obtained by adding a morphism $x : 1 \to I$ to $\mathbb{C}$. Such construction captures the notion of parameterisation: regarding $\mathbb{C}$ as a theory (objects = types, morphisms = terms), the terms of the theory of $\mathbb{C}[x : I]$ have an extra parameter x of type I. This is used in [LS86] to characterise a functional completeness property of cartesian closed categories, meaning that in such categories every term with an extra parameter of type I, *i.e.* a morphism in $\mathbb{C}[x : I]$, can be represented by a term in $\mathbb{C}$ which does not involve x. By generalising this to the fibred setting, we can then characterise semantically functional completeness for polymorphic λ-calculi, where there are two sorts of parameters to consider: type variables and term variables.

In [LS86], it is shown that for cartesian (closed) categories $\mathbb{C}[x : I]$ can be presented as the Kleisli category of a certain comonad on $\mathbb{C}$. Categories with finite products, also called cartesian categories, constitute the basic categorical structure for the interpretation of algebraic theories: terms are given relative to a typing context (for the free variables). Thus, given that types = objects, finite products provide an appropriate structure to interpret contexts. On top of this basic structure, we may require additional features, *e.g.* exponentials to interpret $\longrightarrow$-types.

In generalising the abovementioned result about Kleisli categories to fibrations, we are led to reformulate the problem in 2-categorical terms. We prove it for cartesian objects in a suitable structured 2-category. In order to instantiate this result to the 2-category of fibrations, we prove another important technical result: the existence of Kleisli fibrations for comonads in this 2-category. The construction of these Kleisli fibrations is based on the already mentioned characterisation of fibred adjunctions. It yields as an easy corollary the existence of polynomial fibrations for fibrations with finite products and is shown to be also appropriate for fibrations bearing the structure required to interpret polymorphic λ-calculi. We can then show adequate versions of functional completeness for the calculi $\lambda \rightarrow$ and $\lambda\omega$. Among the immediate applications in programming language semantics, we outline an interpretation of some aspects of ML-style module features, following [FP92].

The structure of the thesis is as follows: ' §n' refers to Chapter 'n', ' §n.m' refers to section 'm' in Chapter 'n', and similarly for subsections and items therein.

§1 reviews the basic material on fibred categories. 2-categories play an organisational role in this thesis and thus a few basic concepts of this theory are included right at the start. We continue with the basic notions about fibrations, organising them into 2-categories $\mathcal{F}ib(\mathbb{B})$, for fibrations with base $\mathbb{B}$, and $\mathcal{F}ib$ for fibrations over arbitrary bases. We review the correspondence between fibrations and in-dexed categories, as well as with internal categories. We also describe some fibred structure necessary to interpret certain type theories, as given in the following chapter.

In §2 we review the categorical interpretation of intuitionistic propositional and first-order predicate calculus, and of polymorphic λ-calculi. We also recall the definition of logical predicates for the simply typed λ-calculus. The chapter concludes with some auxiliary results about reflective and coreflective categories, used to analyse some of the examples in §4.3.

§3 contains the main technical results about fibred adjunctions. In §3.1 we analyse some 2-categorical aspects of the change-of-base construction, which result in a series of algebraic laws concerning fibred-2-cells presented in §3.1.1. In §3.2 we prove the fundamental property relating fibred adjunctions and change-of-base, Theorem 3.2.3. It yields two important corollaries: 3.3.6 and 3.3.11. The first of these gives a new and simple proof of a well-known characterisation of fibred limits, as given in [Gra66, BGT91]. The second gives the categorical counterpart of logical predicates for simply typed λ-calculus.

§4 makes explicit the connection between the lifting of cartesian closed structure through a fibration, with appropriate structure, with logical predicates via a suitable internal language. The so-called 'Basic Lemma' for logical predicates, which is their essential property, is shown in this context to be a consequence of the soundness of typing for the interpretation of simply typed λ-calculus in a ccc, again by recourse to the internal language of the fibration involved. We present a few examples of fibrations with structure, which admit the interpretation of logical predicates, like admissible subsets for $\omega\mathcal{C}po$ and Kripke logical predicates, among other examples of fibred categories whose cartesian closed structure can be inferred from the abovementioned Corollary 3.3.11. We then make a few comparative remarks between our approach to logical predicates and that of [MR91]. We conclude the chapter with a categorical characterisation of the induction principle for inductively defined data types in a distributive category. The latter are the basis of the Charity programming system [CS91]. The formulation of the abovementioned induction principle exploits the view of a fibred category as a category of predicates, as in the case of logical predicates.

§5 develops the technical results required to carry out the abovementioned presentation of polynomial fibrations as Kleisli fibrations for comonads. In §5.3, we review the notion of comonad and Kleisli object in a 2-category. After recalling the appropriate notion of finite products in a 2-category in §5.3.1, we introduce cartesian objects in a 2-category in §5.3.2, and make explicit some of their intrinsic structure. These objects are a direct generalisation of categories with finite products. Under suitable assumptions on the 2-category, we prove that cartesian objects with an indeterminate can be presented as Kleisli objects for a comonad, thereby generalising the abovementioned result of Lambek to the 2-categorical level. §5.4.1 presents the construction of Kleisli objects for comonads in $\mathcal{F}ib(\mathbb{B})$, which is a simple generalisation of that for categories. It also shows how the so-called simple fibration of [Jac91a] can be presented as a Kleisli fibration. In §5.4.2 we present the construction of Kleisli objects for comonads in $\mathcal{F}ib$, Theorem 5.4.11, based on the results of the previous section and a fattorisation for oplax cocones in $\mathcal{F}ib$, using the algebraic properties of fibred 2-cells developed in §3.1.1.

§6 applies the results of the previous chapter to build polynomial fibrations, which interpret the two-level parameterisation of polymorphic λ-calculi. Thus we can characterise so-called contextual and functional completeness properties of these calculi. In §6.3.1 we show how these constructions can be applied to interpret some features of the ML module system: signatures, structures and functors.

§7 contains concluding remarks and considerations for further work.

Chapter 1

Preliminaries on fibrations

This chapter introduces the basic concepts of fibrations or fibred categories. Fibred categories capture the notion of a category varying (continuously) over another. As such, they form the structure required to interpret predicate logics, where predicates correspond to variable propositions, indexed by the type contexts of their free variables. Similarly, fibrations provide a setting to interpret polymorphic calculi, where the terms, or functions, are indexed by the type variables occurring in them. Hence, such terms can be interpreted as morphisms of a fibred category, over a category of contexts, where a context declares the types of the variables which may occur free in an expression.

Our presentation of this preliminary material follows mostly [Jac91a]. We include the material relevant to the applications in this thesis. We consider fibrations from a logical viewpoint, with our application to the interpretation of languages as primary. The categorical interpretation of certain logics will be reviewed in §2.1, once the appropriate technical notions have been introduced in the present chapter. Of course, fibred category theory goes beyond this logical/type theoretic application; see [Bén85] for the foundational relevance of fibrations for category theory.

We assume the reader is familiar with the basic concepts of category theory, as in [Mac71]. We require some basic concepts of 2-categories, which we review in the next section. In §1.2 we recall some basic definitions concerning fibrations. In §1.3, we review an alternative formulation of variable categories, in terms of *indexed categories*; we recall the equivalence between fibrations and indexed categories given by the Grothendieck construction. §1.4 presents structure for fibrations relevant to the interpretation of languages like polymorphic λ-calculi and first-order logic. §1.5 presents elementary notions of internal categories and their relationship with fibrations.

1.0.1 Notational Convention.

- Categories will generally be written $\mathbb{A}, \mathbb{B}$, etc.

- $\mathcal{S}et$ denotes the category of sets and functions, relative to a given universe, as in [Mac71, p.21]. $\mathcal{C}at$ denotes the 2-category of small categories, functors and natural transformations.

- Composition of morphisms, functors, *etc.* is expressed by $\circ$ or juxtaposition, so that $f \circ g$ and fg denote the composite of $f : B \to C$ and $g : A \to B$.

- Given a category $\mathbb{C}$, we denote the product of two objects A and B by $A \times B$, with associated projections $\pi_{A,B} : A \times B \to A$ and $\pi'_{A,B}$:

$A \times B \to B$. Similarly, $A + B$ denotes the sum of the two objects, with associated injections $\iota_{A,B} : A \to A + B$ and $\iota'_{A,B} : A \to A + B$. We sometimes omit subscripts for brevity. Given $f : A \to C$ and $g : B \to C$, their pullback is written

$$\begin{tikzcd}
A \underset{f,g}{\times} B \arrow[r, "g^*(f)"] \arrow[d, "f^*(g)"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

We sometimes write $f^\star(B)$ for $A \underset{f,g}{\times} B$. Given a span $(s : D \to A, t : D \to B)$ such that $f \circ s = g \circ t$, we write $\langle s, t \rangle : D \to A \underset{f,g}{\times} B$ for the unique mediating morphism. Also, we may write $\hat{t}$ for $\langle s, t \rangle$ whenever convenient.

- For categories $\mathbb{C}$ and $\mathbb{D}$, the category of functors from $\mathbb{C}$ to $\mathbb{D}$ and natural transformations between them will be written $\mathbb{D}^{\mathbb{C}}$ or $[\mathbb{C}, \mathbb{D}]$.

- $\mathbb{C}^{op}$ denotes the dual category of $\mathbb{C}$, obtained by reversing the morphisms.

- For a category $\mathbb{C}$, $|\mathbb{C}|$ denotes its class (or set) of objects.

- We write $\alpha : F \overset{\cdot}{\to} G : \mathbb{C} \to \mathbb{D}$ (or briefly $\alpha : F \overset{\cdot}{\to} G$ or even $\alpha : F \Rightarrow G$) for a natural transformation between the functors $F, G : \mathbb{C} \to \mathbb{D}$.

1.1 2-categorical preliminaries

We present basic definitions concerning 2-categories to the extent we need them when dealing with fibred adjunctions in §3. This material is from [KS74], where further references can be found.

1.1.1. Definition (2-category). A *2-category* $\mathcal{K}$ consists of:

- objects or 0-cells $A, B, \ldots$
- morphisms or 1-cells $f : A \to B, \ldots$
- 2-cells $\alpha : f \Rightarrow g, \ldots$
- The objects and morphisms form a category $\mathcal{K}_0$, called the *underlying category* of $\mathcal{K}$.
- For objects A and B, the morphisms $A \to B$ and the 2-cells between them form a category $\mathcal{K}(A, B)$ under *vertical composition*, denoted $\beta \circ \alpha$ or simply $\beta\alpha$. The identity 2-cell on $f : A \to B$ is denoted by 1_f. $\mathcal{K}(A, B)$ is referred to as a hom-category.
- There is an operation of *horizontal composition* of 2-cells, whereby from 2-cells

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\gamma"] & C \end{tikzcd}$$

we get

$$A \underset{vg}{\overset{uf}{\genfrac{}{}{0pt}{}{\longrightarrow}{\longrightarrow}}} C \quad \Downarrow \gamma \star \alpha$$

Under this operation the 2-cells form a category with identities

$$A \underset{1_A}{\overset{1_A}{\genfrac{}{}{0pt}{}{\longrightarrow}{\longrightarrow}}} A \quad \Downarrow 1_{1_A}$$

- In the situation

$$\begin{tikzcd} A \arrow[r, bend left=50, "f"] \arrow[r, phantom, shift left=2, "\Downarrow\alpha"] \arrow[r] \arrow[r, phantom, shift right=2, "\Downarrow\beta"] \arrow[r, bend right=50, "h"'] & B \arrow[r, bend left=50, "u"] \arrow[r, phantom, shift left=2, "\Downarrow\gamma"] \arrow[r] \arrow[r, phantom, shift right=2, "\Downarrow\delta"] \arrow[r, bend right=50, "w"'] & C \end{tikzcd}$$

we have the following *interchange law*

$$(\delta \star \beta) \circ (\gamma \star \alpha) = (\delta \circ \gamma) \star (\beta \circ \alpha)$$

and for any pair of composable 1-cells f and g

$$1_g \star 1_f = 1_{gf}$$

We usually write $f\alpha g$ for $1_f \star \alpha \star 1_g$. By the interchange law, this is the only kind of horizontal composition we need. Observe that a 2-cell in a 2-category has *vertical* domain and codomain given by 1-cells and *horizontal* domain and codomain given by the 0-cells which constitute the usual domain and codomain for the 1-cells involved. We will display such information as $\alpha : f \Rightarrow g : A \to B$, or more simply $\alpha : f \Rightarrow g$, to indicate that α is a 2-cell from f to g, where f and g are 1-cells (the vertical domain and codomain respectively) from A to B (the horizontal domain and codomain respectively).

The paradigmatic 2-category is $\mathcal{C}at$, whose objects are small categories, 1-cells are functors and 2-cells are natural transformations. The reason for introducing 2-categories is that we need to consider not only $\mathcal{C}at$ but also other 2-categories. In particular, for any small category $\mathbb{A}$, the slice category $\mathcal{C}at/\mathbb{A}$ is again a 2-category, with 2-cells those natural transformations $\alpha : H \dot{\to} K : F \to G$, where F and G are functors into $\mathbb{A}$, such that $G_\alpha = 1_F$. Similarly, $\mathcal{C}at^{\to}$, the category of functors and commuting squares is a 2-category with 2-cells being pairs of natural transformations (α', α) as displayed below

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, shift left=2, "H'"] \arrow[r, shift right=2, "K'"'] \arrow[r, phantom, "\downarrow\alpha'"] \arrow[d, "F"'] & \mathbb{D} \arrow[d, "G"] \\
\mathbb{A} \arrow[r, shift left=2, "H"] \arrow[r, shift right=2, "K"'] \arrow[r, phantom, "\downarrow\alpha"] & \mathbb{B}
\end{tikzcd}
$$

i.e. $G\alpha' = \alpha F$. When considering fibrations, we will need sub-2-categories of $\mathcal{C}at/\mathbb{A}$ and $\mathcal{C}at^{\rightarrow}$.

1.1.2. Definition. For a 2-category $\mathcal{K}$, $\mathcal{K}^{op}$ is the 2-category obtained from it by reversing the direction of the morphisms but not the 2-cells, and $\mathcal{K}^{co}$ is the 2-category obtained from $\mathcal{K}$ by reversing the direction of the 2-cells but not the 1-cells. In terms of hom-categories:

$$
\begin{aligned}
\mathcal{K}^{op}(A, B) &= \mathcal{K}(B, A) \\
\mathcal{K}^{co}(A, B) &= (\mathcal{K}(B, A))^{op}
\end{aligned}
$$

Note that $(\mathcal{K}^{op})^{co} = (\mathcal{K}^{co})^{op}$.

The extra structure present in a 2-category, the 2-cells, makes it possible to define categorical concepts involving equations between natural transformations. A typical case is that of an adjunction.

1.1.3. Definition (Adjunction in a 2-category). An *adjunction* $f \dashv g : B \rightarrow A$ in $\mathcal{K}$ consists of 1-cells $f : A \rightarrow B$ and $g : B \rightarrow A$ together with 2-cells $\eta : 1_A \Rightarrow gf$ and $\epsilon : fg \Rightarrow 1_B$ satisfying $(\epsilon f) \circ (f\eta) = 1_f$ and $(g\epsilon) \circ (\eta g) = 1_g$.

We write the data for such an adjunction as $f \dashv g : B \to A$ via η, ϵ. η is called the *unit* and ϵ the *counit* of the adjunction.

The equations between 2-cells are expressible as

$$
\begin{tikzcd}
B \arrow[rr, "1_B"] \arrow[dr, "g"'] & & B \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow\eta"] & \\
& A \arrow[ur, "f"] \arrow[rr, "1_A"'] \arrow[u, phantom, "\Uparrow\epsilon"] & & A
\end{tikzcd}
\quad = \quad 1_g
$$

$$
\begin{tikzcd}
A \arrow[rr, "1_A"] \arrow[dr, "f"'] & & A \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\
& B \arrow[ur, "g"] \arrow[rr, "1_B"'] \arrow[u, phantom, "\Downarrow\eta"] & & B
\end{tikzcd}
\quad = \quad 1_f
$$

1.1.4. Remark. An adjunction $f \dashv g : B \to A$ via η, ϵ in $\mathcal{K}$ becomes $g \dashv f : A \to B$ via ϵ, η in $\mathcal{K}^{co}$ and $f \dashv g : A \to B$ via η, ϵ in $\mathcal{K}^{op}$.

Clearly, in $\mathcal{C}at$, this definition yields the standard notion of adjunction between categories. The case when both η and ϵ are isomorphisms is the 2-categorical notion of *equivalence*. Similarly, we may define a map between adjunctions in a 2-category, as in $\mathcal{C}at$ in [Mac71, p.97].

1.1.5. Definition (Map of adjunctions). Given adjunctions $f \dashv g : B \to A$ and $f' \dashv g' : B' \to A'$ a map from $f \dashv g$ to $f' \dashv g'$ consists of a pair of 1-cells $(l : A \to A', k : B \to B')$ such that $l \circ g = g' \circ k$, $k \circ f = f' \circ l$ and either of the following two equivalent conditions hold:

$$
\begin{aligned}
l\eta &= \eta' l \qquad (1.0)\\
k\epsilon &= \epsilon' k \qquad (1.1)
\end{aligned}
$$

To see that the equations above are equivalent, we use a simple 'pasting' argument. We show (1.0) implies (1.1): (1.0) amounts to

$$
\begin{tikzcd}
A \arrow[rr, "1_A"] \arrow[dr, "f"'] & & A \arrow[r, "l"] & A' \\
& B \arrow[ur, "g"'] \arrow[r, "k"'] \arrow[u, phantom, "\Downarrow\eta"] & B' \arrow[ur, "g'"'] &
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
A \arrow[r, "l"] \arrow[dr, "f"'] & A' \arrow[rr, "1_{A'}"] \arrow[dr, "f'"'] & & A' \\
& B \arrow[r, "k"'] & B' \arrow[ur, "g'"'] \arrow[u, phantom, "\Downarrow\eta'"] &
\end{tikzcd}
$$

Then adjoining ϵ on the LHS and ϵ' on the RHS of both diagrams above and using the adjunction laws we get

$$
\begin{tikzcd}
& A \arrow[rr, "l"] & & A' \arrow[dr, "f'"] \arrow[d, phantom, "\Downarrow\epsilon'"] & \\
B \arrow[ur, "g"] \arrow[rr, "k"'] & & B' \arrow[ur, "g'"] \arrow[rr, "1_{B'}"'] & & B'
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
& A \arrow[rr, "1_A"] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & \arrow[d, phantom, "\Downarrow\eta"] & A \arrow[r, "l"] & A' \arrow[dr, "f'"] \arrow[d, phantom, "\Downarrow\epsilon'"] & \\
B \arrow[ur, "g"] \arrow[rr, "1_B"'] & & B \arrow[ur, "g"'] \arrow[r, "k"'] & B' \arrow[ur, "g'"] \arrow[rr, "1_{B'}"'] & & B'
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
& A \arrow[r, "l"] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & A' \arrow[rr, "1_{A'}"] \arrow[dr, "f'"'] & \arrow[d, phantom, "\Downarrow\eta'"] & A' \arrow[dr, "f'"] \arrow[d, phantom, "\Downarrow\epsilon'"] & \\
B \arrow[ur, "g"] \arrow[rr, "1_B"'] & & B \arrow[r, "k"'] & B' \arrow[ur, "g'"] \arrow[rr, "1_{B'}"'] & & B'
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
& A \arrow[r, "l"] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & A' \arrow[dr, "f'"] & \\
B \arrow[ur, "g"] \arrow[rr, "1_B"'] & & B \arrow[r, "k"'] & B'
\end{tikzcd}
$$

and thus (1.1) holds. The other direction of the equivalence is obtained by duality.

Finally, we introduce morphisms between 2-categories, namely 2-functors, and 2-natural transformations between them.

1.1.6. DEFINITION (2-functor, 2-natural transformation). A *2-functor* $F : \mathcal{K} \to \mathcal{L}$ between 2-categories $\mathcal{K}$ and $\mathcal{L}$ sends objects of $\mathcal{K}$ to objects of $\mathcal{L}$, 1-cells of $\mathcal{K}$ to 1-cells of $\mathcal{L}$ and 2-cells of $\mathcal{K}$ to 2-cells of $\mathcal{L}$, preserving domains, codomains, compositions and identities.

A *2-natural transformation* $\eta : F \Rightarrow F'$ between 2-functors $F, F' : \mathcal{K} \to \mathcal{L}$ assigns to each object A of $\mathcal{K}$ a morphism $\eta_A : FA \to F'A$ in $\mathcal{L}$, such that for every $f : A \to B$ in $\mathcal{K}$

$$\eta_B \circ Ff = F'f \circ \eta_A$$

and for every 2-cell $\alpha : f \Rightarrow g$ in $\mathcal{K}$

$$\begin{tikzcd} FA \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] \arrow[r, phantom, "\Downarrow F\alpha"] & FB \arrow[r, "\eta_B"] & F'B \end{tikzcd} = \begin{tikzcd} FA \arrow[r, "\eta_A"] & F'A \arrow[r, bend left, "F'f"] \arrow[r, bend right, "F'g"'] \arrow[r, phantom, "\Downarrow F'\alpha"] & F'B \end{tikzcd}$$

The functors *cod*, *dom* : $\mathcal{C}at^{\to} \to \mathcal{C}at$ taking $F : \mathbb{A} \to \mathbb{B}$ to $\mathbb{B}$ and $\mathbb{A}$ respectively, with a similar action on morphisms and 2-cells, are examples of 2-functors. The natural transformation $\alpha : dom \to cod$ whose component at $F : \mathbb{A} \to \mathbb{B}$ is F is then a 2-natural transformation, by definition of 2-cells in $\mathcal{C}at^{\to}$.

Just like a functor between categories preserves commutative diagrams, a 2-functor preserves commutative diagrams of 1-cells and 2-cells, since it preserves all kinds of composition and identities. In particular, a 2-functor $F : \mathcal{K} \to \mathcal{L}$ maps adjunctions and equivalences in $\mathcal{K}$ to $\mathcal{L}$.

With the above definitions we have the 2-category 2-$\mathcal{C}at$ of 2-categories, 2-functors and 2-natural transformations. So, it is clear what a 2-adjunction between 2-categories means. In particular, $(\_)^{op} : \mathcal{C}at \to \mathcal{C}at^{co}$, which sends a category to its dual, is a 2-isomorphism.

We say a 2-category $\mathcal{K}'$ is a sub-2-category of $\mathcal{K}$ if its underlying category $\mathcal{K}'_0$ is a subcategory of $\mathcal{K}_0$, and for every pair of objects A, B of $\mathcal{K}'$, the hom-category $\mathcal{K}'(A, B)$ is a subcategory $\mathcal{K}(A, B)$. Of course, horizontal and vertical composition and identities in $\mathcal{K}'$ are as in $\mathcal{K}$.

Universal constructions in 2-categories have a 2-dimensional aspect. For instance, consider the following pullback in $\mathcal{C}at$

$$\begin{tikzcd}
\mathbf{B} \underset{F,G}{\times} \mathbf{C} \arrow[r, "G^*(F)"] \arrow[d, "F^*(G)"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbf{C} \arrow[d, "G"] \\
\mathbf{B} \arrow[r, "F"'] & \mathbf{A}
\end{tikzcd}$$

Given objects $I \in |\mathbb{B}|$ and $J \in |\mathbb{C}|$ such that $FI = GJ$, there is a unique object, written $\langle I, J \rangle$ of $\mathbb{B} \underset{F,G}{\times} \mathbb{C}$ such that $F^*(G)\langle I, J \rangle = I$ and $G^*(F)\langle I, J \rangle = J$. The 2-dimensional aspect is that for morphisms $f : I \to I'$ in $\mathbb{B}$ and

$g : J \to J'$ in $\mathbb{C}$ with $Ff = Gg$, there is a unique morphism $\langle f, g\rangle : \langle I, J\rangle \to \langle I', J'\rangle$ such that $F^*(G)\langle f, g\rangle = f$ and $G^*(F)\langle f, g\rangle = g$. This is formulated 2-categorically as follows: for any span of functors $\mathbb{B} \xleftarrow{I} \mathbb{D} \xrightarrow{J} \mathbb{C}$ such that $F \circ I = G \circ J$, there is a unique functor $\langle I, J\rangle : \mathbb{D} \to \mathbb{B} \underset{F,G}{\times} \mathbb{C}$ such that $F^*(G) \circ \langle I, J\rangle = I$ and $G^*(F) \circ \langle I, J\rangle = J$. And for 2-cells $\alpha : I \Rightarrow I'$ and $\beta : J \Rightarrow J'$ with $F\alpha = G\beta$, there is a unique 2-cell $\langle \alpha, \beta\rangle : \langle I, I'\rangle \to \langle J, J'\rangle$ such that $F^*(G)\langle \alpha, \beta\rangle = \alpha$ and $G^*(F)\langle \alpha, \beta\rangle = \beta$. We will use this pairing notation throughout.

1.2 Basic fibred concepts

This section reviews basic notions about fibrations. Only a few illustrative examples will be given. Others will appear in the applications and more can be found in [Jac91a], from where we borrow most of the material in the remaining of the chapter.

The notion of fibration or fibred category, introduced in [Gro71], captures the concept of a category varying over, or indexed by, another category. Before giving the definition, we recall the analogous situation for sets. A family $\{X_i\}_{i\in I}$ of sets indexed by a set I is a function $X : I \to \mathcal{S}et$. We may regard this as a 'set' X varying over I. It can be equivalently presented as a function $p : \overline{X} \to I$, since such a function gives rise to the family $\{X_i = p^{-1}(i)\}_{i\in I}$ and conversely, given a family $\{X_i\}_{i\in I}$ we get $p : \coprod_{i\in I} X_i \to I$, where $\coprod_{i\in I} X_i$ is the disjoint union of the X_i's and p maps an element in X_i to i. These constructions between morphisms into I and

I-indexed families are mutually inverse. We can summarise this situation by the following isomorphism:

$$\mathcal{S}et/I \cong \mathcal{S}et^I$$

where $\mathcal{S}et/I$ denotes the usual slice category of morphisms into I and commutative triangles, and $\mathcal{S}et^I$ is the category of functors from I, regarded as a discrete category, to $\mathcal{S}et$. These equivalent views of indexed families of sets have their categorical counterparts: a function $X : I \to \mathcal{S}et$ is generalised to an indexed category, *cf.* Definition 1.3.1, while a function $p : X \to I$ is generalised to a fibration, *cf.* Definition 1.2.1. The above isomorphism becomes an equivalence between fibred and indexed categories, *cf.* Proposition 1.3.6 below. Despite this equivalence, the notion of fibration is technically more convenient, as is forcibly argued in [Bén85].

1.2.1. Definition (Fibrations and cofibrations). Consider afunctor $p : \mathbb{E} \to \mathbb{B}$.

(i) A morphism $f : X \to Y$ in $\mathbb{E}$ is (p-)*cartesian* (over a morphism $u = pf : A \to B$ in $\mathbb{B}$) if for every $f' : X' \to Y$ with $pf' = u \circ v$ in $\mathbb{B}$, there exists a unique morphism $\phi_{f'}, : X' \to X$ such that $p\phi_{f'} = v$ and $f' = f \circ \phi_f$. Diagrammatically,

$$
\begin{tikzcd}
 & X' \arrow[d, dashed, "\exists!\phi_{f'}"'] \arrow[dr, "f'"] & \\
\mathbb{E} \arrow[ddd, dotted, "p"] & X \arrow[r, "f"'] & Y \\
 & pX' \arrow[d, "v"'] \arrow[dr, "pf'"] & \\
 & A \arrow[r, "u"'] & B \\
\mathbb{B} & &
\end{tikzcd}
$$

Thus, a cartesian morphism f is a 'terminal lifting' of u. We call such f a *cartesian lifting* of u. In general, when $pf = u$ we say f is above or over u.

(ii) Dually, a morphism $g : X \to Y$ is (p-) *cocartesian* (over a morphism $u = pg : A \to B$ in $\mathbb{B}$) if for every $g' : X \to Y'$ with $pg' = w \circ u$ in $\mathbb{B}$, there exists a unique morphism $\psi_{g'} : Y \to Y'$ such that $p\phi_{g'} = w$ and $g' = \psi_{g'} \circ g$.

(iii) The functor $p : \mathbb{E} \to \mathbb{B}$ is called a *fibration* if for every $X \in |\mathbb{E}|$ and $u : A \to pX$ in $\mathbb{B}$, there is a cartesian morphism with codomain X, such that its image along p is u. $\mathbb{B}$ is then called the *base* of the fibration and $\mathbb{E}$ its *total category*. Dually, p is a *cofibration* if $p^{op} : \mathbb{E}^{op} \to \mathbb{B}^{op}$ is a fibration, *i.e.* for every $X \in |\mathbb{E}|$ and $u : pX \to B$ in $\mathbb{B}$ there is a cocartesian morphism with domain X above u. If p is both a fibration and a cofibration, it is called a *bifibration*.

(iv) For $A \in |\mathbb{B}|, \mathbb{E}_A$, the *fibre* over A, denotes the subcategory of $\mathbb{E}$ whose objects are above A and its morphisms, called (p-)*vertical*, are above 1_A.

1.2.2. Examples. We now introduce three important examples of fibrations. The first motivates terminology concerning fibrations. These examples will be used throughout to illustrate various concepts.

Family fibration The following standard construction of a fibration over $\mathcal{S}et$ is described in [Bén85]. It provides a simple understanding of some fibred conepts. Every category $\mathbb{C}$ gives rise to a *family fibration* $f(\mathbb{C}) : \mathit{Fam}(\mathbb{C}) \to \mathcal{S}et$. Objects of $\mathit{Fam}(\mathbb{C})$ are families $\{X_i\}_{i \in I}$ of $\mathbb{C}$-objects, I a set, *i.e.* a mapping $X : I \to |\mathbb{C}|$; morphisms $(u, \{f_i\}_{i \in I}) : \{X_i\}_{i \in I} \to \{Y_j\}_{j \in J}$ are pairs consisting of a function $u : I \to J$ (in $\mathcal{S}et$) and a family of morphisms such that $f_i : X_i \to Y_{u(i)}$ in $\mathbb{C}$. $f(\mathbb{C})$ takes a family of objects to its indexing set and a morphism to its first component. $(u, \{f_i\}_{i \in I})$ is cartesian when every f_i is an isomorphism. $f(\mathbb{C})$ is then a fibration since given $u : I \to J$ and $\{Y_i\}_{j \in J}$, $(u, \{1_{Y_{u(i)}}\}) : \{Y_{u(i)}\}_{i \in I} \to \{Y_j\}_{j \in J}$ is cartesian above u.

Codomain fibration For any category $\mathbb{C}$, consider the functor $\mathit{cod} : \mathbb{C}^{\to} \to \mathbb{C}$, where $\mathbb{C}^{\to}$ is the category of morphisms of $\mathbb{C}$, *i.e.* $\mathbb{C}^{\to}$ is the functor category $[0 \to 1, \mathbb{C}]$ and $0 \to 1$ denotes the category with two objects and one morphism between them. The functor cod takes $f : A \to B$ to B and (h, k) to k. A cartesian morphism for cod is a pullback square. Thus, whenever $\mathbb{C}$ has pullbacks, *cod* is a fibration. Note that for $A \in |\mathbb{C}|$, the fibre over A is simply the slice category $\mathbb{C}/A$. When there is more than one category under consideration, we write $cod_{\mathbb{C}} : \mathbb{C}^{\rightarrow} \to \mathbb{C}$.

Subobject fibration A related example of fibration is the following. Given a category $\mathbb{C}$, let $Sub(\mathbb{C})$ be the full subcategory of $\mathbb{C}^{\rightarrow}$ whose objects are subobjects in $\mathbb{C}$, *i.e.* equivalence classes of monos. Let $\imath : Sub(\mathbb{C}) \to \mathbb{C}$ be the restriction of *cod* to $Sub(\mathbb{C})$. Caresian morphisms are as for *cod*. When $\mathbb{C}$ has pullbacks of monos along arbitrary morphisms, $\imath$ is a fibration. The fibre over A is the preorder category of subobjects of A.

This fibration plays a fundamental role in categorical logic, since it is the one that determines what the internal logic of the category $\mathbb{C}$ is. That is, the logical connectives, quantifiers and so on which we can interpret in $\mathbb{C}$, regarding predicates as subobjects, depends on the structure of the subobject or internal logic fibration. This remark will become clear when we review some basics of categorical logic in §2.1. Some concrete examples will be analysed later in §4.3.

As immediate consequence of the definition of cartesian morphisms, we have the following proposition:

1.2.3. PROPOSITION. *Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathbb{E}$.* Then,

- *If f and g are cartesian, so is $g \circ f$.*

- *If g and $g \circ f$ are cartesian, so is f.*

For $p = cod : \mathbb{C}^{\rightarrow} \rightarrow \mathbb{C}$, the above proposition yields the following standard result about pullbacks.

1.2.4. Corollary. *Consider the following commutative diagram in $\mathbb{C}$, where P and Q name the corresponding squares*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "P"] & Y \arrow[r, "g"] \arrow[d, "j"] \arrow[dr, phantom, "Q"] & Z \arrow[d, "k"] \\
X' \arrow[r, "l"'] & Y' \arrow[r, "m"'] & Z'
\end{tikzcd}$$

- *If P and Q are pullbacks, so is the outer rectangle.*

- *If Q and the outer rectangle are pullbacks, so is P.*

1.2.5. Remark. Proposition 1.2.3 allows to give an alternative definition of fibration: consider a functor $p : \mathbb{E} \rightarrow \mathbb{B}$

- a morphism $f : Y \rightarrow X$ is v-cartesian if for any $h : Z \rightarrow X$ with $ph = pf$, there exists a unique vertical $\phi : Z \rightarrow Y$ such that $f \circ \phi = h$

- p is a fibration if for every $X \in |\mathbb{E}|$ and $u : A \rightarrow pX$, there exists a

v-cartesian morphism $f : Y \to X$, and the composite of two v-cartesian morphisms is v-cartesian.

If p is a fibration, a morphism is v-cartesian iff it is cartesian. Thus both definitions of fibration agree.

If a funcor $p : \mathbb{E} \to \mathbb{B}$ is a fibration, we display it as $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$. A choice of a cartesian morphism for every appropriate morphism in $\mathbb{B}$ is called a *cleavage* for p which is then a *cloven* fibration, and denoted by $\overline{(\_)}$, so that for $u : I \to pX$ in $\mathbb{B}$, $\overline{u}(X) : u^*(X) \to X$ denotes the chosen cartesian morphism above u. We occasionally need to add the fibration p as extra superscript to the cleavage, when there is more than one fibration under consideration.

Assuming the Axiom of Choice, it is always possible to give a cleavage for a fibration. We thus implicitly assume that the fibrations we deal with are cloven. As we see in §3, this assumption allows elegant algebraic formulations and proofs of properties of fibrations. Most properties we consider admit an intrinsic formulation using cartesian morphisms, independent of a choice of cleavage. It is clear in the proofs that the property at hand is independent of the chosen cleavage.

1.2.6. Definition. Let $\overline{(\_)}$ be a cleavage for $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$.

- For $u : I \to J$ in $\mathbb{B}$, $\overline{(\_)}$ determines a *reindexing functor* $u^* : \mathbb{E}_J \to \mathbb{E}_I$ as follows:

– on objects $X \mapsto u^*(X)$, where $u^*(X)$ denotes the domain of the cartesian lifting of u with codomain X (given by $\overline{(\_)}$)

– on morphisms, for $f : X \to Y$ in $\mathbb{E}_J$, $u^*(f)$ is determined as the unique vertical map making the following diagram commute

$$
\begin{tikzcd}
u^*(X) \arrow[r, "\overline{u}(X)"] \arrow[d, "u^*(f)"'] & X \arrow[d, "f"] \\
u^*(Y) \arrow[r, "\overline{u}(Y)"'] & Y
\end{tikzcd}
$$

using the fact that $\overline{u}(Y)$ is cartesian. This assignment of morphisms is functorial by the universal property of cartesian morphisms.

- For every $I \in |\mathbb{B}|$, there is an isomorphism $\gamma_I : 1_{\mathbb{E}_I} \stackrel{\cdot}{\to} 1^*_{pI}$ determined by the universal property of the morphisms $\overline{1_I}(X)$. For every pair of composable maps $u : J \to K$ and $v : I \to J$ in $\mathbb{B}$, there is an isomorphism $\delta_{v,u} : v^* \circ u^* \stackrel{\cdot}{\to} (u \circ v)^*$ determined by the cartesian morphisms $\overline{u \circ v}(X)$.

The above natural isomorphisms satisfy coherence conditions, induced by the cartesian morphisms. Such conditions occur explicitly in the definition of an indexed category, Definition 1.3.1 and so we do not repeat them here. When such isomorphisms are actual identities, the fibration is called *split* and the corresponding cleavage is a *splitting*. When the γ's are identities, the cleavage is *normalised*. Without loss of generality, we may assume that cleavages are normalised. For the dual notion of cocleavage for a cofibration we denote by $\underline{u}(Y) : Y \to u_!(Y)$ the chosen cocartesian lifting of $u : pY \to B$, which determines a coreindexing functor $u_! : \mathbb{E}_{pY} \to \mathbb{E}_B$. Different cleavages for the same fibration give rise to different, but naturally isomorphic, reindexing functors.

For a functor $p : \mathbb{E} \to \mathbb{B}$, given a morphism $u : A \to B$ in $\mathbb{B}, X \in |\mathbb{E}_A|$ and $Y \in |\mathbb{E}_B|$, let

$$\mathbb{E}_u(X, Y) = \{f : X \to Y \text{ in } \mathbb{E} \mid pf = u\}$$

1.2.7. Proposition. *Let $p : \mathbb{E} \to \mathbb{B}$ be a functor, $u : A \to B$ a morphism in $\mathbb{B}$, $X \in |\mathbb{E}_A|$ and $Y \in |\mathbb{E}_B|$.*

- *(i) If p is a fibration then $\mathbb{E}_u(X, Y) \cong \mathbb{E}_A(X, u^*(Y))$ (naturally in X and Y).*
- *(ii) If p is a cofibration then $\mathbb{E}_u(X, Y) \cong \mathbb{E}_{\mathbb{B}}(u_!(X), Y)$ (naturally in X and Y).*
- *(iii) If p is a fibration then*

p is a cofibration iff for every $u : A \to B$ in $\mathbb{B}, u^* : \mathbb{E}_B \to \mathbb{E}_A$ has a left adjoint

Proof. (i) and (ii) are straightforward consequences of the definition of cartesian and cocartesian morphisms respectively. For (iii),

$$\mathbb{E}_A(X, u^*(Y)) \cong \mathbb{E}_u(X, Y) \cong \mathbb{E}_B(u_!(X), Y)$$

which means that the coreindexing functors are left adjoints to the corresponding reindexing functors, *i.e.* $u_! \dashv u^{*}* : \mathbb{E}_B \to \mathbb{E}_A$, where $u_! : \mathbb{E}_A \to \mathbb{E}_B$ is determined dually to u^*. $\square$

We now characterise the property of a functor being a fibration in terms of the existence of a cleavage for it. This is taken from [Gra66], where it is called the *Chevalley criterion*.

1.2.8. PROPOSITION. *Given a functor $p : \mathbb{E} \to \mathbb{B}$, consider the pullback square*

$$\begin{tikzcd}
\mathbb{E} \underset{p,cod}{\times} \mathbb{B}^{\to} \arrow[r, "cod_B^*(p)"] \arrow[d, "p^*(cod_B)"'] & \mathbb{B}^{\to} \arrow[d, "cod_B"] \\
\mathbb{E} \arrow[r, "p"'] & \mathbb{B}
\end{tikzcd}$$

Let $I_p = \langle cod_{\mathbb{E}}, p^{\to} \rangle : \mathbb{E}^{\to} \to \mathbb{E} \underset{p,cod}{\times} \mathbb{B}^{\to}$ be the unique mediating functor into the pullback induced by the square

$$\begin{tikzcd}
\mathbb{E}^{\to} \arrow[r, "p^{\to}"] \arrow[d, "cod_E"'] & \mathbb{B}^{\to} \arrow[d, "cod_B"] \\
\mathbb{E} \arrow[r, "p"'] & \mathbb{B}
\end{tikzcd}$$

p is a fibration iff I_p has a right-adjoint-right-inverse, i.e. the counit of the adjunction is the identity.

Proof. I_p maps $f : X \to Y$ in $\mathbb{E}$ to (Y, pf). We simply record that a right-adjoint right-inverse Cl to I_p amounts precisely to a cleavage for p; it assigns to every pair $(X, u : I \to pX)$ a cartesian morphism $Cl(Y, u)$ above u. □

1.2.9. Remark. The above formulation of fibration can be used to give a 2-categorical notion of fibration, *i.e.* when a morphism p in a 2-category is a fibration, generalising the situation in $\mathcal{C}at$. Of course, the above formulation makes sense in 2-categories with appropriate structure. See [Str73] for details.

1.2.1 The 2-categories of fibrations

We now define morphisms between fibrations and 2-cells between them. These notions organise fibrations into 2-categories $\mathcal{F}ib(B)$ for fibrations over a given base $\mathbb{B}$, and $\mathcal{F}ib$ for fibrations over arbitrary bases. These 2-categories give a framework in which we can define structure for fibrations, especially in terms of adjunctions.

1.2.10. Definition (Fibred 1-cells and 2-cells). Given $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ and $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{A}\end{smallmatrix}$, a morphism $(\tilde{K}, K) : p \to q$ is rven by a commutative square

$$\begin{tikzcd} \mathbb{E} \arrow[r, "\tilde{K}"] \arrow[d, "p"'] & \mathbb{D} \arrow[d, "q"] \\ \mathbb{B} \arrow[r, "K"'] & \mathbb{A} \end{tikzcd}$$

where $\tilde{K}$ preserves cartesian morphisms, meaning that if f is p-cartesian, $\tilde{K}f$ is q-cartesian. $(\tilde{K}, K)$ is called a *fibred 1-cell* and $\tilde{K}$ a *fibred functor* over K; it determines a collection of functors $\{\tilde{K}|_A : \mathbb{E}_A \to \mathbb{D}_{KA}\}$ between the corresponding fibres. Any pair of cleavages $\overline{(\_)^p}, \overline{(\_)^q}$ determines, for every $u : A \to B$, a natural isomorphism

$$\phi_u : \tilde{K}|_A \circ u^{*\ p} \xrightarrow{\sim} (Ku)^{*\ q} \circ \tilde{K}|_B$$

satisfying: for $u : A \to B$, $v : B \to C$

$$\begin{aligned} \phi_{1_A} \circ \tilde{K}|_A \ \gamma_A &= \gamma_{KA} \tilde{K}|_A \\ \phi_{v \circ u} \circ \tilde{K}|_A \ \delta^p_{u,v} &= \delta^q_{Ku,Kv} \tilde{K}|_C \circ (Ku)^{*\ q} \phi_v \circ \phi_u v^{*\ p} \end{aligned}$$

Given fibred 1-cells $(\tilde{K}, K), (\tilde{L}, L) : p \to q$, a *fibred 2-cell* from $(\tilde{K}, K)$ to $(\tilde{L}, L)$ is a pair of natural transformations $(\tilde{\sigma} : \tilde{K} \dot{\to} \tilde{L}, \sigma : K \dot{\to} L)$ with $\tilde{\sigma}$ above σ, meaning that $q\tilde{\sigma}_X = \sigma_{pX}$ for every $X \in \mathbb{E}$. We display such a fibred 2-cell as follows

$$\begin{tikzcd}
\mathbf{E} \arrow[r, bend left, "\tilde{K}"] \arrow[r, bend right, "\tilde{L}"'] \arrow[r, phantom, "\Downarrow\tilde{\sigma}"] \arrow[d, "p"'] & \mathbf{D} \arrow[d, "q"] \\
\mathbf{B} \arrow[r, bend left, "K"] \arrow[r, bend right, "L"'] \arrow[r, phantom, "\Downarrow\sigma"] & \mathbf{A}
\end{tikzcd}$$

and we write it as $(\tilde{\sigma}, \sigma) : (\tilde{K}, K) \Rightarrow (\tilde{L}, L)$.

In this way we have a 2-category $\mathcal{F}ib$, with fibrations as objects, fibred 1-cells and fibred 2-cells, with the evident compositions inherited from $\mathcal{C}at$. Dually, we have a 2-category $\mathcal{C}o\mathcal{F}ib$ of cofibrations, cofibred functors and cofibred 2-cells.

1.2.11. Examples.

- A functor $F : \mathbb{C} \to \mathbb{D}$ induces a $\mathcal{S}et$-fibred functor $Fam(F) : Fam(\mathbb{C}) \to Fam(\mathbb{D})$ by $\{X_i\}_{i\in I} \mapsto \{FX_i\}_{i\in I}$. Analogously, a natural transformation $\alpha : F \Rightarrow G$ induces a $\mathcal{S}et$-fibred 2-cell $Fam(\alpha) : Fam(F) \Rightarrow Fam(G)$, $Fam(\alpha)_{\{X_i\}} = \{\alpha_{X_i}\}_{i\in I}$. We thus have a 2-functor $Fam(F) : \mathcal{C}at \to \mathcal{F}ib(\mathcal{S}et)$.

- Consider a functor $F : \mathbb{C} \to \mathbb{D}$ such that both $\mathbb{C}$ and $\mathbb{D}$ have and F preserves pullbacks. The induced functor between the categories of morphisms $F^{\to} : C^{\to} \to \mathbb{D}^{\to}$ is a fibred functor over F between the respective codomain fibrations of $\mathbb{C}$ and $\mathbb{D}$. Thus,

$$(F^{\to}, F) : (cod_C : \mathbb{C}^{\to} \to \mathbb{C}) \to (cod_D : \mathbb{D}^{\to} \to \mathbb{D})$$

is a fibred 1-cell. Given another pullback-preserving functor $G : \mathbb{C} \to \mathbb{D}$, any natural transformation $\gamma : F \dot{\to} G$ induces a fibred 2-cell $(\alpha^{\to}, \alpha)$: $(F^{\to}, F) \Rightarrow (G^{\to}, G)$, where for $h : X \to Y \in |\mathbb{C}^{\to}| \alpha^{\to}_h$ is

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Fh"'] & GX \arrow[d, "Gh"] \\ FY \arrow[r, "\alpha_Y"'] & GY \end{tikzcd}$$

Instantiating the notion of adjunction in a 2-category (Definition 1.1.3) in $\mathcal{F}ib$, we obtain the following notion of fibred adjunction.

1.2.12. Definition. Given $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ and $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{A}\end{smallmatrix}$, a *fibred adjunction* between them is given by pair of fibred 1-cells $(\tilde{F}, F) : p \to q$ and $(\tilde{G}, G) : q \to p$ together with a pair of fibred 2-cells $(\tilde{\eta}, \eta) : (1_{\mathbb{E}}, 1_{\mathbb{B}}) \Rightarrow (\tilde{G} \circ \tilde{F}, G \circ F)$ and $(\tilde{\epsilon}, \epsilon) : (\tilde{F} \circ \tilde{G}, F \circ G) \Rightarrow (1_{\mathbb{D}}, 1_{\mathbb{A}})$ such that

(i) $\tilde{F} \dashv \tilde{G} : \mathbb{D} \to \mathbb{E}$ via $\tilde{\eta}, \tilde{\epsilon}$ (in $\mathcal{C}at$)

(ii) $F \dashv G : \mathbb{A} \to \mathbb{B}$ via η, ϵ (in $\mathcal{C}at$)

(iii) p and q constitute a map of adjunctions between the two above, *i.e.* $p\tilde{\eta} = \eta p$

(or equivalently $q\tilde{\epsilon} = \epsilon q$)

Such a fibred adjunction is displayed by

$$\begin{tikzcd}
\mathbb{E} \arrow[r, shift left, "\tilde{F}"] \arrow[d, "p"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, shift left, "\tilde{G}"] \arrow[d, "q"] \\
\mathbb{B} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{A} \arrow[l, shift left, "G"]
\end{tikzcd}$$

When the components of $\tilde{\eta}$ and $\tilde{\epsilon}$ are cartesian and the square (fibred 1-cell) $(\tilde{F}, F) : p \to q$ is a pullback, we call it a *cartesian fibred adjunction*. This terminology is justified by Theorem 3.2.3.

1.2.13. Remark. For a cartesian fibred adjunction, the adjoint transpose of a cartesian morphism $f : \tilde{F}X \to Y$ in $\mathbb{D}$, which is $f^\vee \triangleq \tilde{G}f \circ \tilde{\eta}_X$, is again cartesian. This is equivalent to the cartesianness of the components of $\tilde{\eta}$.

Although the notion of subfibration does not play a major role in this thesis, we include its definition to make sense of a few statements below and in §3.

1.2.14. Definition.

(i) Given a fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ and a subcategory $\mathbb{E}'$, $J : \mathbb{E}' \to \mathbb{E}, p \circ J : \mathbb{E}' \to \mathbb{B}$ is a *subfibration* of p if, for every object $X \in |\mathbb{E}|$, if $f : Y \to JX$ is cartesian in $\mathbb{E}$, then f is in $\mathbb{E}'$.

(ii) More generally, given fibrations $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ and $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{A}\end{smallmatrix}$, where $\mathbb{A}$ is a subcategory of $\mathbb{B}$, $J : \mathbb{A} \to \mathbb{B}$, we say q is a *subfibration* of p if q is a subfibration of

$J^*(p)$ in the sense of (i).

Since $\mathcal{F}ib$ is a sub-2-category of $\mathcal{C}at^{\to}$, we get by restriction of *cod* : $\mathcal{C}at^{\to} \to \mathcal{C}at$ the 2-functor *cod* : $\mathcal{F}ib \to \mathcal{C}at$, which maps every fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ to its base category $\mathbb{B}$. We know that *cod* : $\mathcal{C}at^{\to} \to \mathcal{C}at$ is a fibration (*cf.* Example 1.2.2). The following proposition shows that its restriction to $\mathcal{F}ib$ is still a fibration — a subfibration in fact, since cartesian morphisms for it are pullback squares.

1.2.15. PROPOSITION. *Given a fibration $q : \mathbb{D} \to \mathbb{A}$ and an arbitrary functor $K : \mathbb{B} \to \mathbb{A}$, consider a pullback diagram*

$$\begin{tikzcd}
\mathbb{E} \arrow[r, shift left=2, "\tilde{F}"] \arrow[r, phantom, "\perp"] \arrow[d, "p"'] & \mathbb{D} \arrow[l, shift left=2, "\tilde{G}"] \arrow[d, "q"] \\
\mathbb{B} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathbb{A} \arrow[l, shift left=2, "G"]
\end{tikzcd}$$

$K^*(q)$ is a fibration, with a morphism f in $K^*(\mathbb{D})$ being $K^*(q)$-cartesian iff $q^*(K)(f)$ is q-cartesian. The above diagram is therefore a morphism of fibrations.

Proof. An elegant proof is in [Gra66], using the characterisation of fibrations given in Proposition 1.2.8. In elementary terms, given an object of $K^*(\mathbb{D})$, determined by a pair of compatible objects $\langle I \in \mathbb{B}, X \in \mathbb{D}\rangle$, and a morphism $u : J \to I$ in $\mathbb{B}$, its cartesian lifting $\overline{(u)}^{K^*(q)}(I, X)$ is determined by u and the cartesian lifting $\overline{(Ku)}^q(I, X) : Ku^*(X) \to X$. $\square$

We say that $K^*(q)$ is obtained from q by *change of base* along K. We assume that the cleavage for $K^*(q)$ is obtained from that of q as in the above proof. So $q^*(K)$ preserves cleavages. If q has a splitting, so does $K^*(q)$.

The fibre of *cod* : $\mathcal{F}ib \to \mathcal{C}at$ over a category $\mathbb{A}$ is the 2-category $\mathcal{F}ib(\mathbb{A})$, consisting of fibrations with base $\mathbb{A}$. Morphisms $F : p \to q$ are functors between the total categories of p and q which commute with the fibrations ($qF = p$) and preserve cartesian morphisms. Such an F is called a ($\mathbb{A}$)-fibred functor, in preference to the usual terminology of 'cartesian functor'. 2-cells are natural transformations $\alpha : F \ \dot{\to}\ G : p \to q$ such that $q\alpha = p$. Such an α is called a *vertical* natural transformation or an $\mathbb{A}$-fibred 2-cell. We use the prefix $\mathbb{A}$- to denote 2-categorical concepts in $\mathcal{F}ib(\mathbb{A})$ to distinguish them from the corresponding ones in $\mathcal{F}ib$. We may thus speak of an $\mathbb{A}$-fibred adjunction. Usually we drop the prefix when the context makes it clear which 2-category is meant. We will also refer $\mathbb{A}$-fibred concepts as *vertical*.

Considering only split fibrations and splitting-preserving morphisms, we have sub-2-categories $\mathcal{F}ib_{sp}$ and $\mathcal{F}ib(\mathbb{A})_{sp}$.

1.2.16. Remark. In view of Proposition 1.2.15, we may regard a fibred 1-cell $(\tilde{K}, K) : p \to q$, with $K : \mathbb{A} \to \mathbb{B}$, as an $\mathbb{A}$-fibred 1-cell $\hat{K} = \langle p, \tilde{K} \rangle : p \to K^*(q)$.

Using Proposition 1.2.8, we have the following characterisation of morphisms in $\mathcal{F}ib(\mathbb{A})$

1.2.17. PROPOSITION. *The data* $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{A}\end{array}$, $\begin{array}{c}\mathbb{D}\\ \downarrow q\\ \mathbb{A}\end{array}$ *and* $F : p \to q$ *in* $\mathcal{C}at/A$ *induces the following commutative square*

$$\begin{tikzcd}
\mathbb{E}^{\to} \arrow[r, "F^{\to}"] \arrow[d, "I_p"'] & \mathbb{D}^{\to} \arrow[d, "I_q"] \\
\mathbb{E} \underset{p,cod}{\times} \mathbb{A}^{\to} \arrow[r, "cod^*(F)"'] & \mathbb{D} \underset{q,cod}{\times} \mathbb{A}^{\to}
\end{tikzcd}$$

where $cod^*(F) : cod^*(p) \to cod^*(q)$ *is uniquely determined by* F *and the pullbacks* $\mathbb{E} \underset{p,cod}{\times} \mathbb{A}^{\to}$ *and* $\mathbb{D} \underset{p,cod}{\times} \mathbb{A}^{\to}$. *Given right-adjoint-right-inverses* $\overline{(-)}^p$ *and* $\overline{(-)}^q$ *for* I_p *and* I_q *(with units* η_p *and* η_q*) respectively, the square above induces a canonical natural transformation* $\gamma : F^{\to} \circ \overline{(-)}^p \dot{\to} \overline{(-)}^q \circ cod^*(F), \gamma = \eta_q F^{\to} \overline{(-)}^p$. *Then,* F *preserves cartesian morphisms iff* γ *is an isomorphism.*

Proof. For an object $(Y, u : I \to pY)$ of $\mathbb{E} \underset{q,cod}{\times} \mathbb{A}^{\to}$

$$\gamma_{(Y,u:I\to pY)} : F(u^{*p}(Y)) \to u^{*q}FY$$

is the canonical vertical morphism determined by $\overline{(u)}^q(FY)$. Hence F preserves Cartesian morphisms iff every such vertical morphism is an isomorphism. □

When γ in the above proposition is the identity, F preserves cleavages.

1.3 Indexed categories and the Grothenieck construction

We continue our review with indexed categories, which are sometimes more intuitive than fibrations and help in understanding topics such as fibred adjunctions and fibred comonads, as developed in §3 and §5.

Indexed categories are essentially equivalent to fibrations but technically often less convenient. For instance, it is easy to prove that the composition of two fibrations is again a fibration, but this cannot be expressed directly for indexed categories. See [Bén85] for furth er relevant discussion. More importantly, the notion of fibration makes sense in any 2-category [Str73, Joh92].

Recalling the analogy between fibrations and families of sets in §1.2, the isomorphism

$$\mathcal{S}et/I \cong \mathcal{S}et^I$$

leads us to consider another version of a varying category, *i.e.* a category varying continuously over another, as a generalisation of $\mathcal{S}et^I$ as the category of I-indexed families of sets. Now, the indexing object is not a mere set I but a category and similarly, the indexed objects X_i are not just sets but categories.

1.3.1. Definition (Indexed category). Given a category $\mathbb{B}$, a $\mathbb{B}$-*indexed category* is a pseudo-functor $\mathcal{F}\colon \mathbb{B}^{op} \to \mathcal{C}at$; it is given by the following data

- For every object $A \in |\mathbb{B}|$, a category $\mathcal{F}A$.

- For every morphism $f : A \to B$ in $\mathbb{B}$, a functor $f^* : \mathcal{F}B \to \mathcal{F}A$, together with natural isomorphisms $\gamma_A : 1_{\mathcal{F}A} \cong 1_A^*$ and $\delta_{f,g} : (f^* \circ g^*) \cong (g \circ f)^*$ satisfying the following coherence conditions: for $u : A \to B, v : B \to C$ and $w : D \to A$ in $\mathbb{B}$

$$
\begin{array}{rcl}
\delta_{u,1_B} \circ u^*\gamma_B & = & 1_{u^*} \\
\delta_{1_A,u} \circ \gamma_A u^* & = & 1_{u^*} \\
\delta_{w,v\circ u} \circ w^*\delta_{u,v} & = & \delta_{u\circ w,v} \circ \delta_{w,u}v^* : w^* \circ u^* \circ v^* \ \dot{\to}\ (v \circ u \circ w)^*
\end{array}
$$

1.3.2. Remark. The coherence conditions above express associativity and identity laws. Their role is clear in Proposition 1.3.6.(iii). Often these isomorphisms are identities, in which case we have a *strict* indexed category, *i.e.* a functor $\mathcal{F} : \mathbb{B}^{op} \to \mathcal{C}at$.

1.3.3. Examples. The following examples of strict indexed categories are taken from [BGT91]. They are basic to the area of algebraic specifications.

(i) (*Many-sorted sets*) Consider the following functor $\mathcal{SS} : \mathcal{S}et^{op} \to \mathcal{C}at$

$$
\begin{array}{rcl}
\mathcal{SS}(I) & = & \mathcal{S}et^I \\
\mathcal{SS}(f : I \to J) & = & (X : J \to \mathcal{S}et) \mapsto (X \circ f : I \to \mathcal{S}et)
\end{array}
$$

The objects of a, fibre $\mathcal{SS}(I)$ are families of sets. The functor $\mathcal{SS}(f : I \to J)$ performs reindexing along f. The coherent isomorphisms are identities.

(ii) (*Many-sorted algebraic signatures*) Consider the functor $(\_)^+ : \mathcal{S}et \to \mathcal{S}et$, which assigns to a set S the free semigroup it generates, *i.e.* the set S^+ of all finite non-empty sequences of elements of S. The functor $\mathcal{AS} = \mathcal{SS} \circ ((\_)^+)^{op} : \mathcal{S}et^{op} \to \mathcal{C}at$ is an indexed category; its fibres $\mathcal{AS}(S)$ are S-sorted algebraic signatures, *i.e.* they consist of, for every non-empty sequence $s_1, \ldots, s_n \in S^+$ regarded as arity or rank $s_1, \ldots, s_{n-1} \to s_n$, a set of operation symbols of that rank. A reindexing functor $\mathcal{AS}(f : S \to S')$ transforms S'-sorted signatures into S-sorted signatures, renaming sorts according to f.

1.3.4. Definition. Let $\mathcal{F} : \mathbb{B}^{op} \to \mathcal{C}at$ and $\mathcal{G} : \mathbb{B}^{op} \to \mathcal{C}at$ be indexed categories.

- An *indexed functor* $\mathcal{H} : \mathcal{F} \to \mathcal{G}$ consists of:

  (i) For every $A \in |\mathbb{B}|$, a functor $\mathcal{H}(A) : \mathcal{F}(A) \to \mathcal{G}(A)$
  (ii) For every $u : A \to B$, a natural isomorphism

$$\phi_u : \mathcal{G}(u) \circ \mathcal{G}(B) \xrightarrow{\sim} \mathcal{H}(A) \circ \mathcal{F}(u)$$

  satisfying coherence conditions with the γ's and δ's of Definition 1.3.1; *cf.* Definition 1.2.10 where these conditions are given for the equivalent concept, of fibred 1-cells.

- An *indexed natural transformation* $\alpha : \mathcal{H} \Rightarrow \mathcal{H}' : \mathcal{F} \to \mathcal{G}$, consists of a natural transformation $\alpha_A : (\mathcal{A}) \stackrel{\cdot}{\to} \mathcal{H}'(A)$ for every object $A \in |\mathbb{B}|$, such that for every $u : A \to B, \mathcal{G}(u)\alpha_{\mathbb{B}} = \alpha_A \mathcal{F}(u)$, modulo the ϕ_u's.

Indexed categories over a given category $\mathbb{B}$, indexed functors and indexed natural transformations form a 2-category $\mathcal{ICat}(\mathbb{B})$, with the evident fibrewise notions of composition and identities, inherited from $\mathcal{C}at$.

1.3.5. Remark. Having defined indexed functors and indexed natural transformations, the notion of indexed adjunction is then analogous to the standard notion of adjunction between categories. We can give the following description, which the reader might find intuitive: given indexed functors $\mathcal{H} : \mathcal{F} \to \mathcal{G}$ and $\mathcal{H}' : \mathcal{G} \to \mathcal{F}$ over $\mathbb{B}, \mathcal{H}$ is an indexed left adjoint to $\mathcal{H}'$ iff:

- For every $A \in |\mathbb{B}|, \mathcal{H}_A \dashv \mathcal{H}'_A$.
- $u : A \to B$, the pair $(\mathcal{F}(u), \mathcal{G}(u))$ preserves the adjunctions, *i.e.* it is a (pseudo-)map of adjunctions from $\mathcal{H}_B \dashv \mathcal{H}'_B$ to $\mathcal{H}_A \dashv \mathcal{H}'_A$. The latter means a map of adjunctions, where the relevant squares commute only up to a given (coherent) isomorphism.

We now show the correspondence between cloven fibrations and indexed categories, due to Grothendieck, which amounts to an 'equivalence' between the 2-categories $\mathcal{F}ib(\mathbb{B})$ and $\mathcal{ICat}(\mathbb{B})$.

1.3,6. Proposition.

(i) *Every cloven fibration* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *gives rise to an indexed category* $\mathcal{F}_p : \mathbb{B}^{op} \to \mathcal{C}at$.

(ii) *Every indexed category* $F : \mathbb{B}^{op} \to \mathcal{C}at$ *gives rise to a fibration* $p_{\mathcal{F}} : \mathcal{GF} \to \mathbb{B}$.

(iii) *The above correspondences set up an 'equivalence' of 2-categories*

$$\mathcal{ICat}(\mathbb{B}) \simeq \mathcal{Fib}(\mathbb{B})$$

so that $\mathcal{F}_{p_{\mathcal{F}}} \simeq \mathcal{F}$ *and* $p_{\mathcal{F}_p} \simeq p$.

Proof.

(i) Given a cloven fibration $p : \mathbb{E} \to \mathbb{B}$, we obtain an indexed category $\mathcal{F}_p : \mathbb{B}^{op} \to \mathcal{C}at$ as follows:

- For every $A \in |\mathbb{B}|$, $\mathcal{F}_p A = \mathbb{E}_A$.
- For every $u : A \to B$, a cleavage $\overline{(\_)}$ induces a reindexing functor $u^* : \mathbb{E}_B \to \mathbb{E}_A$ as given in Definition 1.2.6. As we mentioned then, the universal property of cartesian morphisms uniquely determines natural isomorphisms $\delta_{v,u} : v^* \circ u^* \stackrel{\sim}{\Rightarrow} (u \circ v)^*$ and $\gamma_I : 1_{\mathbb{E}_I} \stackrel{\sim}{\Rightarrow} 1_I^*$, which satisfy the coherence conditions in Definition 1.3.1.

(ii) Given an indexed category $\mathcal{F} : \mathbb{B}^{op} \to \mathcal{C}at$ we define the total category $\mathcal{GF}$, consisting of:

Objects: $\langle A, a\rangle \in |\mathcal{GF}|$ iff $A \in |\mathbb{B}|$ and $a \in |\mathcal{F}A|$. That is (using a hopefully self-explanatory dependent sum notation)

$$|\mathcal{GF}| = \Sigma A : \mathbb{B}.\mathcal{F}A$$

Morphisms: $\langle f, g\rangle : \langle A, a\rangle \to \langle B, b\rangle$ iff $f : A \to B$ in $\mathbb{B}$ and $g : a \to f^*(b)$ in $\mathcal{F}A$. That is

$$\mathcal{GF}(\langle A, a\rangle, \langle B, b\rangle) = \Sigma f : \mathbb{B}(A, B).\mathcal{F}A(a, f^*(b))$$

Identity: $\langle 1_A, \gamma_A(a)\rangle : \langle A, a\rangle \to \langle A, a\rangle$

Composition: Given $\langle f, g\rangle : \langle A, a\rangle \to \langle B, b\rangle$ and $\langle h, j\rangle : \langle B, b\rangle \to \langle C, c\rangle$, let

$$\langle h, j\rangle \circ \langle f, g\rangle = \langle h \circ f, \delta_{f,h}(c) \circ f^*(j) \circ g\rangle$$

The coherence conditions of Definition 1.3.1 are required in orcler to show associativity of composition and the identity laws. The projection functor $p_{\mathcal{F}} : \mathcal{GF} \to \mathbb{B}$ which takes $\langle A, a\rangle$ to A (for objects and morphisms) is then a fibration: for an morphism $u : A \to B$ in $\mathbb{B}$ and an object X in $\mathcal{F}B$, we can choose as cartesian lifting $\bar{u}(X) = \langle u, 1_{u^*X}\rangle$.

(iii) Observe that the fibres of $p_{\mathcal{F}}$ are $\mathcal{GF}_B = \mathcal{F}B$ and the action of the reindexing functors is the same in both fibrations and indexed categories respectively. Any pair of cleavages for a given fibration give rise to equivalent indexers categories.

1.3.7. Remarks.

- The construction of $p_{\mathcal{F}}$ from $\mathcal{F}$ in the above proof is known as the *Grothendieck construction.*

- Dualising the above proposition, we get an analogous result relating cofibrations $p : \mathbb{E} \to \mathbb{B}$ and pseudo-functors $\mathcal{G} : \mathbb{B} \to \mathcal{C}at$.

- The equivalence in the above proposition clearly restricts to one between split fibrations $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ and functors $\mathcal{F} : \mathbb{B}^{op} \to \mathcal{C}at$ (strict indexed categories). Splitting-preserving functors between split fibrations correspond under this equivalence to natural transformations.

1.3.8. Remark. The 2-categorical aspect of the equivalence in Proposition 1.3.6.(iii) implies a correspondence between indexed and fibred adjunctions. Thus, a fibred adjunction $F \dashv G : p \to q$ (between $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ and $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{B}\end{smallmatrix}$) amounts to a family of adjunctions $\{F|_B \dashv G|_B : \mathbb{E}_B \to \mathbb{D}_B\}_{B \in |\mathbb{B}|}$ such that for every $u : B \to B'$, (u^{*p}, u^{*q}) is a (pseudo-)map of adjunctions from $F|_{B'} \dashv G|_{B'}$, to $F|_B \dashv G|_B$.

For an indexed category $\mathcal{F} : \mathbb{B}^{op} \to \mathcal{C}at$ and a functor $H : \mathbb{A} \to \mathbb{B}$, change-of-base of $\mathcal{F}$ along H is given by composition $H^*(\mathcal{F}) = \mathcal{F} \circ H^{op}$. Similarly, for a natural transformation $\alpha : H \stackrel{\cdot}{\to} \mathcal{H}' : \mathbb{A} \to \mathbb{B}$ we have an indexed natural transformation $\mathcal{F}\alpha^{op} : \mathcal{F} \circ (H')^{op} \Rightarrow \mathcal{F} \circ H^{op}$, where $\alpha^{op} : H^{op} \stackrel{\cdot}{\to} (\mathcal{H}')^{op} : \mathbb{A}^{op} \to \mathbb{B}^{op}$ has components $(\alpha^{op})_A = \alpha_A^{op}$.

1.3.9. Definition. The 2-category $\mathcal{ICat}$ has indexed categories $\mathcal{F} : \mathbb{B}^{op} \to \mathcal{C}at$ (over arbitrary categories) as objects. A morphism from $\mathcal{F} : \mathbb{A}^{op} \to \mathcal{C}at$ to $\mathcal{G} : \mathbb{B}^{op} \to \mathcal{C}at$ is given by a functor $H : \mathbb{A} \to \mathbb{B}$ and an indexed functor $\mathcal{H} : \mathcal{F} \to \mathcal{G} \circ H^{op}$; we write $(\mathcal{H}, H)$ for this morphism. A 2-cell $(\tilde{\alpha}, \alpha) : (\mathcal{H}, H) \Rightarrow (\mathcal{H}', H') : \mathcal{F} \to \mathcal{G}$ consists of a natural transformation $\alpha : H \Rightarrow H'$ and an indexed natural transformation $\tilde{\alpha} : \mathcal{G} \circ H^{op} \Rightarrow (\mathcal{G}\alpha^{op}) \circ (\mathcal{G} \circ (H')^{op}$. Compositions and identities are defined using those in $\mathcal{C}at$ and $\mathcal{ICat}(\_)$.

There is a forgetful 2-functor *base*: $\mathcal{ICat} \to \mathcal{C}at$, which takes an indexed category to its base and morphisms and 2-cells to their second components. *base* is a split fibration, with splitting given by composition, as noted above. This observation and Proposition 1.3.6. (iii) yield as immediate consequence the following equivalence.

1.3.10. COROLLARY. *There is a $\mathcal{C}at$-fibred 2-equivalence*

$$\begin{tikzcd}[column sep=tiny] \mathcal{I}\mathcal{C}at \arrow[dr, "base"'] & \equiv & \mathcal{F}ib \arrow[dl, "cod"] \\ & \mathcal{C}at & \end{tikzcd}$$

We regard indexed categories as a convenient means of presenting cloven fibrations. We are not interested in the indexed category itself but in (the total category of) the fibration it yields via the Grothendieck construction.

1.3.11. EXAMPLE. The family fibration $f(\mathbb{C}) : \mathit{Fam}(\mathbb{C}) \to \mathcal{S}et$ results from applying the Grothendieck construction to the (strict) $\mathcal{S}et$-indexed category given by

$$\begin{array}{rcl} I & \mapsto & \mathcal{S}et^I \\ u : I \to J & \mapsto & \_ \circ u : \mathcal{S}et^J \to \mathcal{S}et^I \end{array}$$

We close this section defining the groupoid and the opposite of an indexed category.

1.3.12. DEFINITION. Given $\mathcal{F} : \mathbb{B}^{op} \to \mathcal{C}at$, its *groupoid indexed category* $|\mathcal{F}| : \mathbb{B}^{op} \to \mathcal{C}at$ is defined by

$$\begin{array}{rcl} A & \mapsto & \text{the groupoid subcategory of } \mathcal{F}A, \\ & & \text{consisting only of all the isomorphisms} \\ u : A \to B & \mapsto & \text{the restriction of } \mathcal{F}u \text{ to the groupoid subcategories} \end{array}$$

and its *opposite indexed category* $\mathcal{F}^{\nu op} : \mathbb{B}^{op} \to \mathcal{C}at$ by

$$\begin{array}{rcl} A & \mapsto & (\mathcal{F}A)^{op} \\ u : A \to B & \mapsto & (\mathcal{F}u)^{op} : (\mathcal{F}B)^{op} \to (\mathcal{F}A)^{op} \end{array}$$

1.4 Fibred structure, products and sums

Given a fibration $p : \mathbb{E} \to \mathbb{B}$, its groupoid fibration, written $|p| : Cart(\mathbb{E}) \to \mathbb{B}$, results from applying the Grothendieck construction to the groupoid indexed category of the $\mathbb{B}$-indexed category it induces. It can also be described as the restriction of p to $Cart(\mathbb{E})$, the subcategory of $\mathbb{E}$ consisting of the Cartesian morphisms only. Similarly, the opposite fibration, written $p^{\nu op} : (\mathbb{E}/\mathbb{B})^{\nu op} \to \mathbb{B}$, is obtained by applying the Grothendieck construction to the opposite of its associated indexed category. $p^{\nu op}$ also admits an intrinsic formulation; see [Jac91a, §1.1.11] for details.

Now we can describe fibred structure for $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ in terms of $\mathbb{B}$-fibred adjunctions, as we do in $\mathcal{C}at$ for ordinary categories [Mac71, §V]. The following definitions are from [Jac92].

1.4.1. Definition. A fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ has

- a *fibred terminal object* iff $p : p \to 1_{\mathbb{B}}$ in

$$\begin{tikzcd} \mathbb{E} \arrow[rr, "p"] \arrow[dr, "p"'] & & \mathbb{B} \arrow[dl, "1_{\mathbb{B}}"] \\ & \mathbb{B} & \end{tikzcd}$$

has a fibred right adjoint $1_p : 1_{\mathbb{B}} \to p$, which we call *terminal object functor*, usually written as 1.

- *fibred binary products* iff the diagonal fibred functor $\Delta_p : p \to p \times p$ (where $p \times p$ is the product of p with itself in $\mathcal{F}ib(\mathbb{B})$) has a fibred right adjoint $\times : p \times p \to p$.

- *fibred exponents* (assuming fibred binary products) iff the fibred functor $\langle \pi', \times \rangle : p \times |p| \to |p|^{op} \times p$ (products considered in $\mathcal{F}ib(\mathbb{B})$) obtained by pairing $p \times |p| \hookrightarrow p \times p \xrightarrow{\times} p$ and $p \times |p| \xrightarrow{\pi'} |p| \cong |p|^{op}$, has a fibred right adjoint $exp : |p|^{op} \times p \to p \times |p|$.

A fibration with fibred finite products will be called a *fibred-cc* and one which additionally has fibred exponents a *fibred-ccc*.

1.4.2. Remarks.

- $1_{\mathbb{B}} : \mathbb{B} \to \mathbb{B}$ is a terminal object in $\mathcal{F}ib(\mathbb{B})$. The above definition of fibred terminal object is entirely analogous then to that of a terminal object for an ordinary category. A similar consideration applies to fibred products and exponents.

- The above definitions admit an elementary description in terms of fibrewise structure, preserved by reindexing functors. A fibration has a fibred terminal object iff every fibre has a terminal object and reindexing functors preserve these. Similarly for fibred products and exponents. This correspondence is a consequence of the equivalence between fibrations and indexed categories, Proposition 1.3.6.iii.

- The above notions admit elementary intrinsic definitions, without reference to a cleavage. For instance, $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ has a fibred terminal object if every fibre $\mathbb{E}_I$ has a terminal object $1(I)$ and for any cartesian morphism $f : X \to 1(I), X$ is terminal in $\mathbb{E}_{pX}$.

1.4.3. Examples.

(i) For a category $\mathbb{C}, f(\mathbb{C}) : \mathit{Fam}(\mathbb{C}) \to \mathcal{S}et$ has fibred finite products (respectively exponents) iff $\mathbb{C}$ has finite products (respectively exponents). In one direction, the fibred products/exponents are defined pointwise, *e.g.* $\{X_i\}_{i\in I} \times \{Y_i\}_{i\in I} = \{X_i \times Y_i\}_{i\in I}$. Conversely, $\mathbb{C}$ is the fibre over the one-element set and hence has products (respectively exponents).

(ii) For $\mathbb{C}$ with pullbacks, $\mathit{cod} : \mathbb{C}^{\to} \to \mathbb{C}$ has fibred finite products. For $A \in \mathbb{C}, 1_A$ is a terminal object in $\mathbb{C}/A$ and the product of $f : B \to A$ and $g : C \to A$ is given by the diagonal of their pullback, with projections given by the pullback projections. This is why the pullback is sometimes referred to as 'fibred product'. Preservation of such structure under reindexing is immediate. *cod* is a fibred-ccc when every slice $\mathbb{C}/A$ is a ccc, that is, when

$\mathbb{C}$ is a so-called *locally cartesian closed category* (lccc for short).

(iii) $\imath : Sub(\mathbb{C}) \to \mathbb{C}$ is a sub-fibred-cc of *cod*, *i.e.* the fibred finite products are as given in *cod*. If $\mathbb{C}$ is an elementary topos, then $\imath$ is a fibred-ccc, since in this case every fibre $Sub(\mathbb{C})_A$ is a complete Heyting algebra and hence is cartesian closed (as a poset). See [LS86, Part 11,§5, Exercise 3]

Next, we introduce some indexed products and sums for a fibration. Such structure is necessary for the interpretation of first-order quantifiers in predicate logic, see §2.1.2. The terminology is taken from [Jac91a, §1.5.1,§4.2.1], where a more general form of quantification relative to an arbitrary so-called comprehension category is given. For $\mathbb{A}$ a category with binary products and $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{A}\end{smallmatrix}$, we have the following change-of-base situation for any object I:

$$
\begin{tikzcd}
(\_ \times I)^*(\mathbb{D}) \arrow[r, "q^*(\_ \times I)"] \arrow[d, "q_I"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{D} \arrow[d, "q"] \\
\mathbb{A} \arrow[r, "\_ \times I"'] & \mathbb{A}
\end{tikzcd}
$$

where we have written q_I for $(\_ \times I)^*(q)$.

1.4.4. Definition. Given $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$, $\mathbb{B}$ with binary products and $I \in |\mathbb{B}|$, p has $Cons_I$-products (respectively sums) if both

(i) for every $J \in |\mathbb{B}|$, the reindexing functor $\pi^*_{J,I} : \mathbb{E}_J \to E_{J\times I}$ induced by $\pi_{J,I} : J \times I \to J$ has a right adjoint Π_I (respectively a left adjoint Σ_I) and

(ii) (Beck-Chevalley condition) for every $u : J \to J'$ in $\mathbb{B}$, the canonical natural transformation

$$u^*\Pi_I \stackrel{\cdot}{\to} \Pi_I(u \times 2_I)^*$$

(respectively $\Sigma_I(u \times 1_I)^* \stackrel{\cdot}{\to} u^*\Sigma_I$) is an isomorphism.

p has $Cons_{\mathbb{B}}$-products/sums if it has $Cons_I$-products/sums for every $I \in |\mathbb{B}|$.

Instantiating the general definition of quantification in [Jac91a], we get the following formulation of $Cons_I$-products/sums in terms of fibred adjunctions:

1.4.5. PROPOSITION. *Given* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$, *let* $\delta_I : p \to p_I$ *in* $\mathcal{F}ib(\mathbb{B})$ *be defined as follows: for* $Y \in |\mathbb{E}_J|$,

$$\delta_I Y = \pi^*_{J,I} Y$$

Then, p *has* $Cons_I$*-products (respectively sums) iff* δ_I *has a fibred right (respectively left) adjoint.*

The above proposition explains why we may omit the parameter J in the definition of $Cons_I$-products/sums; we are simply defining one fibred functor as an adjoint.

1.4.6. EXAMPLES.

(i) For $\mathbb{C}$ a category with small products/coproducts, the family fibration $f(\mathbb{C})$ has $Cons_{\mathcal{S}et}$-products/sums. They are given by $\Pi_I(\{X_{(j,i)}\}_{(j,i)\in J\times I} =$

$\{\Pi_{i\in I} X_{(j,i)}\}_{j\in J}$, with an analogous expression for sums. Conversely, if $f(\mathbb{C})$ has $Cons_{\mathcal{S}et}$-products/sums, $\mathbb{C}$ has small products/sums: for a set A, the product/sum of an A-indexed family $\{X_a\}_{a\in A}$ is obtained by applying the right/left adjoint to reindexing along the projection $!_A : A \to 1$.

(ii) $cod : \mathbb{C}^{\to} \to \mathbb{C}$ has $Cons_{\mathbb{C}}$-sums, given by composition: $\Sigma_I(f : A \to J \times I) = \pi \circ f$. In case $\mathbb{C}$ is a lccc, it also has $Cons_{\mathbb{C}}$-products, since in this situation it has right adjoints for every reindexing functor. See [BW90, Theorem 12.4.3].

(iii) Consider $\imath : Sub(\mathcal{S}et) \to \mathcal{S}et$, where we may identify subobjects with subsets. Thus reindexing corresponds to taking inverse images: $u^*(S \subseteq B) = u^{-1}(S) \subseteq A$ (for $u : A \to B$). This fibration has $Cons_{\mathcal{S}et}$-products and sums. They correspond to universal and existential quantification:

$$\begin{array}{rcl} \Pi_I(S \subseteq J \times I) & = & \{j \in J \mid \forall i \in I.\ (j,i) \in S\} \\ \Sigma_I(S \subseteq J \times I) & = & \{j \in J \mid \exists i \in I.\ (j,i) \in S\} \end{array}$$

The Beck-Chevalley condition expresses the interaction between quantification and substitution of terms for free variables. Namely, for $u : J' \to J$ and $S \subseteq J \times I$,

$$\begin{array}{rcl} u^{-1}(\Pi_I(S)) & = & u^{-1}\{j \in J \mid \forall i \in I.\ (j,i) \in S\} \\ & = & \{j' \in J' \mid \forall i \in I.\ (uj',i) \in S\} \\ & = & \{j' \in J' \mid \forall i \in I.\ (j',i) \in (u \times I)^{-1}S\} \\ & = & \Pi_I((u \times I)^{-1}(S)) \end{array}$$

The following proposition shows how $Cons_{\_}$-products are preserved by change-of-base along a finite product preserving functor.

1.4.7. PROPOSITION. *Consider the following change-of-base situation:*

$$\begin{tikzcd} F^*(\mathbf{E}) \arrow[r, "p^*(F)"] \arrow[d, "F^*(p)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E} \arrow[d, "p"] \\ \mathbf{A} \arrow[r, "F"'] & \mathbf{B} \end{tikzcd}$$

Let $\mathbb{A}$ and $\mathbb{B}$ be categories with finite products and F a finite-product preserving functor. If p admits $Cons_{\mathbb{B}}$-products, $F^*(p)$ admits $Cons_{\mathbb{A}}$-products, and the above fibred 1-cell $(p^*(F), F)$ preserves them.

Proof. Observe that for a cartesian projection $\pi_{X,Y} : X \times Y \to X, \pi_{X,Y}^{*F^*(p)} \cong \pi_{FX,FY}^{*p}$ and therefore $\pi_{X,Y}^{*F^*(p)} \dashv \Pi_{FY}$. The Beck-Chevalley condition holds trivially, since F preserves the relevant pullback squares. □

The notion of *generic object* is a key one in the interpretation of impredicative λ-calculi, as in §2.1.3. For instance, it allows us to interpret higher-order impredicative quantification in terms of first-order quantification. The notion of generic object is related to that of representability, as given below.

1.4.8. DEFINITION.

(i) For a category $\mathbb{B}$, an object I determines a fibration, written $dom_I : \mathbb{B}/I \to \mathbb{B}$, with action $J \xrightarrow{f} I \mapsto J$ on objects, being the identity on morphisms. Cartesian liftings are obtained by composition.

(ii) A fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ is *representable* if it is equivalent in $\mathcal{F}ib(\mathbb{B})$ to a fibration of the form $dom_I : \mathbb{B}/I \to \mathbb{B}$.

1.4.9. Remark. When a fibration is such that every fibre is discrete (*i.e.* a set), we call it a *discrete* fibration, like dom_I in the definition above. Note that the fibre $(\mathbb{B}/I)_J = \mathbb{B}(J, I)$. In view of the correspondence between fibrations and indexed categories, a discrete fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ corresponds to a presheaf $F_p : \mathbb{B}^{op} \to \mathcal{S}et$. In particular, dom_I corresponds to the representable presheaf $\mathbb{B}(\_, I)$, which explains the term 'representable' for such fibrations.

Recall that a functor $F : \mathbb{A} \to \mathbb{B}$ is *essentially surjective* if for every object Y of $\mathbb{B}$, there is an object X in $\mathbb{A}$ such that $FX \cong Y$. In particular, an equivalence between categories is essentially surjective.

1.4.10. Definition.

(i) A fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ has a *generic object* if there is a representable fibration $dom_\Omega : \mathbb{B}/\Omega \to \mathbb{B}$ and an essentially surjective fibred functor $ext : dom_\Omega \to p$.

(ii) A fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ has a *strong generic object* if the groupoid fibration $|p| : Cart(\mathbb{E}) \to \mathbb{B}$ is representable.

1.4.11. Remarks.

- In elementary terms, $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ has a generic object if there is an object $G \in |\mathbb{E}|$ such that for every $X \in |\mathbb{E}|$ there is a cartesian morphism $f : X \to G$. Given $ext : dom_\Omega \to p$, we can take G to be $ext(1_\Omega)$ and given any $X \in |\mathbb{E}_I|$, we have an object $x : I \to \Omega$ such that $X \cong ext(x)$ and hence there is cartesian morphism $f : X \cong ext(x) \stackrel{ext(x)}{\to} G$, since $x : x \to 1_\Omega$ is dom_Ω-cartesian and ext is a fibred functor.

- We refer to G itself as the generic object and write $\chi_X : I \to \Omega$ for the underlying morphism pf, so that $X \cong \chi_X^*(G)$ in the above situation. Notice that χ_X with this property need not be unique.

- A representable fibration $dom_\Omega : \mathbb{B}/\Omega \to \mathbb{B}$ has $1_\Omega : \Omega \to \Omega$ as a strong generic object, as does any small fibration, as defined in §1.5 below.

- If $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ has a strong generic object, then it has a generic object: the inclusion $J : |p| \to p$ is essentially surjective. Thus, in elementary terms, having a strong generic object G means that for any object X of $\mathbb{E}_I$ there is a unique $\chi_X : I \to \Omega$ such that $X \cong \chi_X^*(G)$ in $\mathbb{E}_I$.

1.4.12. Examples.

- Let $\mathbb{B}$ be an elementary topos. The subobject fibration $\imath : \mathcal{S}ub(\mathbb{B}) \to \mathbb{B}$ has a strong generic object, namely the subobject classifier $true : 1 \to \Omega$: for any subobject $m : X' \hookrightarrow X$ there is a pullback

$$
\begin{tikzcd}
X' \arrow[r] \arrow[d, hook, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, hook, "true"] \\
X \arrow[r, "\chi_X"'] & \Omega
\end{tikzcd}
$$

- For a category $\mathbb{C}$, the family fibration $f(\mathbb{C})$ has a generic object precisely when $\mathbb{C}$ has a small set of objects. In this case, the strong generic object is $\{x\}_{x\in|\mathbb{C}|}$.

The following result is standard.

1.4.13. PROPOSITION. *Let* $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{array}$ *be a discrete fibration. Let* $F_p : \mathbb{B}^{op} \to \mathcal{S}et$ *be its associated presheaf. The following are equivalent*

(i) $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{array}$ *has a strong generic object.*

(ii) F_p *is representable.*

(iii) $\mathbb{E}$ *has a terminal object.*

Proof: (i) $\Longleftrightarrow$ (ii) is immediate.
(ii) $\Longleftrightarrow$ (iii) Since the fibres of $\mathbb{E}$ are discrete, every morphism in $\mathbb{E}$ is cartesian. Let G be the terminal object of $\mathbb{E}$. pG is a representing object for F_p, since for any object X in $\mathbb{E}_I$, there is a unique cartesian morphism $f : X \to G$, and hence a unique $\chi_X = pf : I \to pG$ such that $X = \chi_X^*(G) = (F_p\chi_X)(G)$. Conversely, if F_p is representable, $\gamma : F_p \cong \mathbb{B}(\_, \Omega), \gamma^{-1}(1_\Omega)$ is a terminal object in $\mathbb{E}$. $\square$

1.5 Internal categories

We end our preliminaries on fibrations by introducing the basic notions of internal categories, which give yet another way of dealing with variable categories. Although internal categories make sense independently of fibrations, the description of some internal concepts is more conveniently expressed in the fibred setting. Besides, it is possible to 'externalise' internal categories to obtain a fibration, and every fibration can be internalised (under certain size conditions) within a presheaf topos, as we show below. Internal categories will not be used in this thesis, but they provide further insight into the way fibrations may arise.

Throughout this subsection $\mathbb{B}$ is assumed to be a category with pullbacks.

1.5.1. Definition (Internal category, functor, natural transformation).

(i) An *internal category* C in $\mathbb{B}$ is given by the following data:

- an *object of objects* $C_0 \in |\mathbb{B}|$;
- an *object of arrows* $C_1 \in |\mathbb{B}|$;
- *domain* and *codomain* morphisms $\delta_0, \delta_1 : C_1 \to C_0$ respectively;
- an *identity* morphism $i : C_0 \to C_1$ such that

$$\delta_0 \circ i = 1_{C_0} = \delta_1 \circ i$$

- a *composition* morphism $c : C_2 \to C_1$ satisfying

$$
\begin{array}{rcll}
\delta_0 \circ c & = & \delta_0 \circ \pi_0 & : C_2 \to C_0 \\
\delta_1 \circ c & = & \delta_1 \circ \pi_1 & : C_2 \to C_0 \\
c \circ (i \underset{1,\delta_0}{\times} 1_{C_1}) & = & \pi_1 & : C_0 \underset{1,\delta_0}{\times} C_1 \to C_1 \\
c \circ (1_{C_1} \underset{\delta_1,1}{\times} i) & = & \pi_0 & : C_1 \underset{\delta_1,1}{\times} C_0 \to C_1) \\
c \circ (c \underset{\delta_1,\pi_1,\delta_0}{\times} 1_{C_1} & = & c \circ (1_{C_1} \underset{\delta_1,\pi_1,\delta_0}{\times} c) & : C_3 \to C_1
\end{array}
$$

where

$$
\begin{tikzcd}
C_2 \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "\delta_0"] \\
C_1 \arrow[r, "\delta_1"'] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_3 \arrow[r, "\pi_1'"] \arrow[d, "\pi_0'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "\delta_0"] \\
C_2 \arrow[r, "\delta_1 \circ \pi_1"'] & C_0
\end{tikzcd}
$$

(ii) An *internal functor* F between two internal categories C and C' consists of a pair of morphisms $F_0 : C_0 \to C_0'$ and $F_1 : C_1 \to C_1'$ in $\mathbb{B}$ satisfying

$$
\begin{array}{rcl}
F_0 \circ \delta_0 & = & \delta_0' \circ F_1 \\
F_0 \circ \delta_1 & = & \delta_1' \circ F_1 \\
F_1 \circ i & = & i' \circ F_0 \\
F_1 \circ c & = & c' \circ (F_1 \underset{\delta_1,\delta_0}{\times} F_1)
\end{array}
$$

(iii) An internal natural transformation $\sigma : F \dashrightarrow G$ between internal functors is a morphism $\sigma : C_0 \to C'_1$ such that

$$
\begin{tikzcd}
 & C_0 \arrow[dl, "F_0"'] \arrow[d, "\sigma"] \arrow[dr, "G_0"] & \\
C'_0 & C'_1 \arrow[l, "\delta'_0"] \arrow[r, "\delta'_1"'] & C'_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[r, "\langle \sigma \circ \delta_0, G_1 \rangle"] \arrow[d, "\langle F_1, \sigma \circ \delta_1 \rangle"'] & C'_2 \arrow[d, "c'"] \\
C'_2 \arrow[r, "c'"'] & C'_1
\end{tikzcd}
$$

The definitions above give the defining data for the 2-category $\mathcal{C}at(\mathbb{B})$ of internal categories in $\mathbb{B}$. Note that $\mathcal{C}at(\mathcal{S}et) = \mathcal{C}at$ that is, an internal category in $\mathcal{S}et$ is simply a small category, the correspondence extending to internal functors and internal natural transformations. Next, we consider some structure pertinent to internal categories.

Let $\mathbb{B}$ be cartesian closed. For $C \in \mathcal{C}at(\mathbb{B})$ and $A \in |\mathbb{B}|$ we have an internal category $C^A = (C_0^A, C_1^A, \delta_0^A, \delta_1^A, i^A, c^A)$, since $(\_)^A : \mathbb{C} \to \mathbb{C}$ is a right adjoint, it preserves the relevant pullbacks in the definition of an internal category; it maps internal categories to internal categories. There is an obvious internal diagonal functor $\Delta : C \to C^A$, given by the adjoint transposes of the projections $\pi : C_i \times A \to C_i$ $(i = 0, 1)$. The following definition is taken from [Jac91a, §1.5, 1.5.4].

1.5.2. Definition. C admits *internal* $Cons_A$*-products* (respectively *sums*) if $\Delta : C \to C^A$ has an internal right (respectively left) adjoint. Internal

$Cons_{\mathbb{B}}$-products/ sums are given by internal $Cons_A$-products/sums for every $A \in |\mathbb{B}|$.

Further structure, such as an internal category having finite products and exponents, an internal ccc, are expressed by rephrasing the definitions for ordinary categories; we just couch them in terms of the 2-category $\mathcal{C}at(\mathbb{B})$ instead of $\mathcal{C}at$.

Now we show how to obtain a fibration from an internal category by a process of *externalisation*. A fibration obtained in this way is called *small*.

1.5.3. Definition (Externalisation).

> Given an internal category $C \in \mathcal{C}at(\mathbb{B})$, let $\Sigma(C)$ be the category with objects (A, X), where $A \in |\mathbb{B}|$ and $X : A \to C_0$, and morphisms $(u, f) : (A, X) \to (B, Y)$ where $u : A \to B$ in $\mathbb{B}$ and $f : A \to C_1$ satisfying $\delta_0 \circ f = X$ and $\delta_1 \circ f = Y \circ u$. Composition and identities in $\Sigma(C)$ are defined from those of $\mathbb{B}$ and the internal ones of C, The first projection $[C] : \Sigma(C) \to \mathbb{B}$ is a split fibration, with cleavage given by composition.

- For $F : C \to D$ in $\mathcal{C}at(\mathbb{B}), [F] : \Sigma(C) \to \Sigma(D)$ is given by

$$(A, X) \mapsto (A, F_0 \circ X) \quad \text{and} \quad (u, f) \mapsto (u, F_1 \circ f).$$

- For $\sigma : F \dot{\to} G$ in $\mathcal{C}at(\mathbb{B}), [\sigma] : [F] \dot{\to} [G]$ has components $[\sigma]_{(A,X)} = (1_A, \sigma \circ X)$.

With the above definitions we get a (2-)full and faithful externalisation 2-functor $[\_] : \mathcal{C}at(\mathbb{B}) \to \mathcal{F}ib(\mathbb{B})$. Such a functor gives a correspondence between internal and fibred 2-categorical concepts, such as those of adjunction and comonad for instance.

Finally, we quote another standard result, [Jac91a, Proposition 1.4.8], which shows that a fibration can be turned into an internal category.

1.5.4. PROPOSITION *(Internalisation). Let* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *be a split fibration, where* $\mathbb{B}$ *is locally small and all fibres are small. Then there is an internal category* $\bar{p}$ *in* $\mathcal{S}et^{\mathbb{B}^{op}}$ *and a change-of-base situation*

$$\begin{tikzcd} \mathbb{E} \arrow[r, "H"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma(\overline{p}) \arrow[d, "{[\overline{p}]}"] \\ \mathbb{B} \arrow[r, "Y"'] & \mathcal{S}et^{\mathbb{B}^{op}} \end{tikzcd}$$

where Y *is the Yoneda embedding.* H *is full and faithful. Furthermore,* p *is a split fibred ccc iff* $\bar{p}$ *is an internal ccc.*

Chapter 2

Preliminaries on categorical logic

This chapter continues the review of preliminary material. We review the categorical interpretation of some type theories which we consider in subsequent chapters. Specifically, we review the interpretation of simply typed λ-calculus and first-order intuitionistic logic in §2.1.1 and §2.1.2 respectively, since we will require them for the categorical account of logical predicates in §4. We also recall the interpretation of polymorphic λ-calculi, used in §6, where we will consider indeterminate elements for the corresponding fibrations.

In §2.2 we recall the definition of logical predicates for applicative structures. The latter are used to give a set-theoretic semantics to simply typed λ-calculus.

In §2.3 we show two simple properties about reflective and coreflective categories, concerning cartesian closure. These are applied in §4.3 to analyse cartesian closed structure in some examples.

2.1 Review of propositional and first-order categorical logic

Categorical logic interprets logics in categories, providing a syntax-free description We recall the basic facts. We review the interpretation of intuitionistic propositional calculus, whose proof-language is the simply typed λ-calculus, and of first-order intuitionistic predicate logic. [LS86] is the basic reference for the former, while for the latter we follow Lawvere's approach [Law70, See83]. With regard to their categorical interpretation, the intuition is that propositions correspond to objects, proofs, or rather proof terms, to morphisms, and the logical connectives conjunction and implication to products and exponentials respectively. As for predicate logic, predicates are indexed propositions and hence objects of a fibred category. In this context, quantifiers are interpreted as adjoints to appropriate reindexing functors. Summing up, for the propositional calculus we need cartesian closed categories while for the predicate calculus we need fibrations with structure, called first-order hyperdoctrines as in [Pit91].

We use a type theoretic formulation of these logics, according to the proposition-as-types paradigm. We present them as type systems, giving the inference rules for the derivation of the corresponding judgements. In both systems, the disjunctive part, $\{\vee, \bot\}$, is left out for simplicity. It can be handled dually to $\{\wedge, \top\}$, using binary coproducts $+$ and an initial object 0.

We also recall the categorical interpretation of polymorphic lambda calculi, following [Jac91a]. These calculi extend the simply typed one by allowing type variables. These variables index the types and terms in which they occur. A categorical setting to interpret these calculi is a fibration with structure.

2.1.1 Intuitionistic propositional calculus - Simply typed λ-calculus

Within the proposition-as-types approach, propositions correspond to types. Proofs of a given proposition from a given set of hypotheses correspond to terms of the respective type, relative to a context corresponding to the hypotheses.

The calculus has three kinds of judgements:

$$P\ \mathsf{Prop} \qquad \Gamma \vdash t : P \qquad \Gamma \vdash t = t' : P$$

which respectively assert that P is a proposition, t is a proof-term of proposition P in context Γ and that t and t' are equal proofs of the same proposition, in the same context. A context is a finite assignment of propositions to variables $[x_1 : P_1, \ldots, x_n : P_n]$, where all the x_i's are different. Γ in the judgements above provides types for the free variables occurring in the terms t and t'. We regard $\vdash$ as entailment: the P_i's are the assumptions and P is the conclusion of the sequent. The empty context is omitted from the lhs of $\vdash$.

There are three groups of inference rules, corresponding to the three kinds of judgements above. The first group deals with the formation of propositions; it ensures that the set of propositions contains a 'true' proposition (unit type) and is closed under conjunction (binary product type constructor) and implication (arrow type constructor):

$$\frac{}{\top\ \mathsf{Prop}} \quad \frac{P\ \mathsf{Prop} \quad Q\ \mathsf{Prop}}{P \times Q\ \mathsf{Prop}} \quad \frac{P\ \mathsf{Prop} \quad Q\ \mathsf{Prop}}{P \longrightarrow Q\ \mathsf{Prop}}$$

The second group deals with the formation of terms for structured propositions

$$\frac{}{\Gamma \vdash !_\Gamma : \top}$$

$$\frac{\Gamma \vdash t : P \quad \Gamma \vdash t' : Q}{\Gamma \vdash \langle t, t' \rangle : P \times Q} \qquad \frac{\Gamma \vdash t : P \times Q}{\Gamma \vdash \pi t : P} \qquad \frac{\Gamma \vdash t : P \times Q}{\Gamma \vdash \pi' t : Q}$$

$$\frac{\Gamma, x : P \vdash t : Q}{\Gamma \vdash \lambda x : P.t : P \longrightarrow Q} \quad \frac{\Gamma \vdash t : P \longrightarrow Q \ \Gamma \vdash t' : P}{\Gamma \vdash tt' : Q}$$

The third group deals with equality of terms of structured propositions

$$\frac{\Gamma \vdash t : \top}{\Gamma \vdash !_\Gamma : \top}$$

$$\frac{\Gamma \vdash t : P \times Q}{\Gamma \vdash t = \langle \pi t, \pi' t \rangle : P \times Q} \qquad \frac{\Gamma \vdash t : P \ \Gamma \vdash t' : Q}{\Gamma \vdash \pi \langle t, t' \rangle = t : P} \qquad \frac{\Gamma \vdash t : P \ \Gamma \vdash t' : Q}{\Gamma \vdash \pi' \langle t, t' \rangle = t' : Q}$$

$$\frac{\Gamma, x : P \vdash t : Q}{\Gamma, x : P \vdash (\lambda x : P.t)x = t : Q} \quad \frac{\Gamma \vdash t : P \longrightarrow Q}{\Gamma \vdash (\lambda x : P.tx) = t : P \longrightarrow Q}$$

A judgement which can be obtained using the inference rules is called *derivable*. The interpretation of this system in a cartesian closed category $\mathbb{C}$ goes as follows:

Derivable judgement	*Interpretation in* $\mathbb{C}$
P Prop	$[\![P]\!] \in \lvert\mathbb{C}\rvert$
$\Gamma \vdash t : P$	$[\![t]\!] : [\![\Gamma]\!] \to [\![P]\!]$
$\Gamma \vdash t = t' : P$	$[\![t]\!] = [\![t']\!]$

and $\Gamma = [x_1 : P_1, \ldots, x_n : P_n]$ is interpreted as $[\![\Gamma]\!] = [\![P_1]\!] \times \ldots \times [\![P_n]\!]$. We outline the interpretation of several inference rules:

- The formation rules for propositions are interpreted by:

$$\begin{array}{rcl} \top\ \mathsf{Prop} & \mapsto & 1(\text{ terminal object }) \\ P \times Q & \mapsto & [\![P]\!] \times [\![Q]\!] \\ P \longrightarrow Q & \mapsto & [\![P]\!] \Rightarrow [\![Q]\!] \end{array}$$

- The formation rules for terms are interpreted using the hom-set isomorphisms of the relevant adjunctions, *e.g.*

$$\frac{\Gamma, x : P \vdash t : Q}{\Gamma \vdash \lambda x : P.t : P \to Q} \mapsto \frac{[\![\Gamma]\!] \times [\![P]\!] \stackrel{[\![t]\!]}{\to} [\![Q]\!]}{[\![\Gamma]\!] \stackrel{\Lambda([\![t]\!])}{\longrightarrow} [\![P]\!] \Rightarrow [\![Q]\!]}$$

  where $\Lambda : \mathbb{C}(A \times B, C) \xrightarrow{\sim} \mathbb{C}(A, B \Rightarrow C)$ is the isomorphism of the exponential adjunction.

- The equalities between terms are seen to hold for their interpretations, using the pertinent adjunction laws. This means that the interpretation in a cartesian closed category $\mathbb{C}$ is sound with respect to the equational theory of the calculus.

A few examples of cartesian closed categories occur in §4.3.

2.1.2 First-order intuitionistic predicate calculus

In the previous calculus there were two main kinds of entities involved: propositions (or types) and terms. The language of the predicate calculus has three kinds of entities: types, predicates and terms. More specifically, we have the following kinds of judgements: the first three correspond to a many-sorted algebraic theory $\mathcal{T}$

$$T\ \mathsf{Type} \quad \Theta \vdash t : T \quad \Theta \vdash t = t' : T$$

where Type is closed under finite products, just like Prop in intuitionistic propositional calculus. In addition, we have the following judgements, corresponding to an intuitionistic predicate logic over the given algebraic theory (predicates $\equiv$ indexed propositions relative to a context)

$$\Theta \vdash P\ \mathsf{Prop} \quad \Theta|\Gamma \vdash p : P \quad \Theta|\Gamma \vdash p = p' : P$$

Γ in the judgements above is a context of propositions $[x_1 : P_1, \ldots, x_n : P_n]$, each one relative to context Θ, *i.e.* $\Theta \vdash P_i\ \mathsf{Prop}$. Thus for a type A, if $[x : A] \vdash P\ \mathsf{Prop}$ then P is simply a predicate with a (potentially) free variable of type A. Sometimes we write $P(x)$ to emphasise the dependence of P on x. $P[x := t]$, sometimes written $P(t)$, denotes the substitution of the term t for x in P.

The inference rules come in three groups. The first group deals with the algebraic theory $\mathcal{T}$; the rules assert that the class of types is closed under finite products and provide the corresponding pairing and projection operations with their associated equations, as done for simply typed λ-calculus. Thus, a function symbol f with arity $T_1, \ldots, T_n, T$ is a term $[x_1 : T_1, \ldots, x_n : T_n] \vdash f : T$. We may write $f(x_1, \ldots, x_n)$ for f to emphasise the dependence on the free variables. The second group of rules deals with predicates and proofs relative to a context. They form an intuitionistic propositional calculus and thus we have the same rules as in §2.1.1 with a type context $\Theta|$ prefixed everywhere, *e.g.*

$$\frac{\Theta|\Gamma, x : P \vdash p : Q}{\Theta|\Gamma \vdash \lambda x : P.p : P \longrightarrow Q}$$

$$\frac{\Theta|\Gamma \vdash p : P \longrightarrow Q \quad \Theta|\Gamma \vdash p' : P}{\Theta|\Gamma \vdash pp' : Q}$$

The third group accounts for the interaction between the theory $\mathcal{T}$ and the predicates. This involves substitution and quantification. For substitution we have

$$\frac{\Theta \vdash t : T \quad [x : T] \vdash P \ \mathsf{Prop}}{\Theta \vdash P[x := t]\mathsf{Prop}}$$

plus rules which express the fact that substitution preserves the propositional connectives and proofs. The formation rules for quantified predicates are

$$\frac{\Theta, x : T \vdash P \ \mathsf{Prop}}{\Theta \vdash \forall x : T.P \ \mathsf{Prop}} \quad \frac{\Theta, x : T \vdash P \ \mathsf{Prop}}{\Theta \vdash \exists x : T.P \ \mathsf{Prop}}$$

The rules handling proofs of such quantified predicates are easily given in 'adjoint' style: we have the following bidirectional rules

$$\frac{\Theta, x : T|\Gamma \vdash p : P}{\Theta|\Gamma \vdash \Lambda x : T.p : \forall x : T.P}$$

$$\frac{\Theta, x : T|\Gamma, h : P \vdash p : Q}{\Theta|\Gamma, h' : \exists x : T.P \vdash p[\langle x, h\rangle := h']Q}$$

The rule for $\exists$ uses a pseudo-substitution notation for the elimination which should not be confused with the usual one for substitution of terms for variables. The associated equations between proof-terms of quantified formulae are such that the above rules yield bijections between the corresponding sets of proof-terms.

For the categorical interpretation of this calculus we need a category $\mathbb{B}$ with finite products to interpret $\mathcal{T}$ (types $\equiv$ objects, terms $\equiv$ morphisms), in the same way as with simply typed λ-calculus in a cartesian closed category. As for predicates, for each context Θ, which corresponds to an object A of $\mathbb{B}$, we need a cartesian closed category $\mathbb{H}_A$ to interpret predicates ($\Theta \vdash P$ Prop) and proofs ($\Theta \mid \Gamma \vdash p : P$) in such context as objects and morphisms respectively, as in §2.1.1. Furthermore, we need for every term in $\mathcal{T}$, corresponding to a morphism $t : C \to D$ in $\mathbb{B}$, a substitution functor $t^* : \mathbb{H}_D \to \mathbb{H}_C$ which preserves the cartesian closed structure. Finally, the 'adjoint' style formulation of the rules for $\forall$ and $\exists$ suggest that quantifiers must be interpreted by functors adjoint to substitution along projections. If $\Theta, x : T \vdash P$ Prop, $\Theta \vdash \forall x : T.P$ is interpreted as $\Pi_T([\![P]\!])$. Here $\pi^*_{A,[\![T]\!]} \dashv \Pi_T : H_{A\times[\![T]\!]} \to H_A$. Dually $\Theta \vdash \exists x : T.P$ is interpreted as $\Sigma_T(P)$, with $\Sigma_T \dashv \pi^*_{A,[\![T]\!]}$. As we mentioned in Example 1.4.6.(iii), the interaction between substitution and quantification imply that such right and left adjoints have to satisfy Beck-

Chevalley conditions. Hence, universal and existential quantifiers correspond to $Cons_{\mathbb{B}}$-products and sums respectively.

The substitution functors give rise then to a $\mathbb{B}$-indexed category, which we can turn into a fibration over $\mathbb{B}$ as shown in §1.3. To sum up, the structure needed to interpret first-order intuitionistic predicate calculus, or rather, the fragment of it we have presented, is the following

2.1.1. Definition. A *first-order hyperdoctrine* is a fibred-ccc $p : \mathbb{H} \to \mathbb{B}$, where $\mathbb{B}$ has finite products, which has $Cons_{\mathbb{B}}$-products and sums.

This definition is a fibred reformulation of Lawvere's hyperdoctrines [Law70], tailored to model first-order rather than higher-order predicate calculus.

2.1.2. Example (Classical set-theoretic models). We have seen in Examples 1.4.3.(iii) and 1.4.6.(iii) that the fibration $\imath : Sub(\mathcal{S}et) \to \mathcal{S}et$ is a first-order hyperdoctrine. The interpretation of first-order intuitionistic logic in it is the classical one: types are sets, terms are functions, predicates are subsets, and connectives and quantifiers have their usual set-theoretic meaning.

Further examples occur in §4.3.

[Pav90] gives a thorough account of higher-order constructive predicate logic in terms of fibrations.

2.1.3 Polymorphic lambda calculi

We review the categorical interpretation of impredicative polymorphic lambda calculi. These calculi generalise the simply typed one by allowing type variables. In addition, type variables may be quantified. They provide a basis for polymorphic programming languages, like ML [MTH90]. We recall three systems: $\lambda \rightarrow, \lambda 2$ and $\lambda\omega$. In these calculi there are three sorts of entities: kinds κ, types τ and terms t. There is a distinguished kind Ω which classifies types. Types in turn classify terms, as in simply typed λ-calculus. There are two levels of contexts: $\Theta = [X_1 : \kappa_1, \ldots , X_n : \kappa_n]$ for kind variables and $\Gamma = [x_1 : \tau_1, \ldots , x_n : \tau_n]$ for term variables.

The judgements areκ Kind

$$\begin{array}{ccc} \kappa\ \mathsf{Kind} & \Theta \vdash \tau : \kappa & \Theta \mid \Gamma \vdash t : \tau \\ & \Theta \vdash \tau = \tau' : \kappa & \Theta \mid \Gamma \vdash t = t' : \tau \end{array}$$

In $\Theta|\Gamma \vdash \ldots$, the types in Γ must be defined with respect to Θ *i.e.* if $x_i : \tau_i$ is in Γ, then $\Theta \vdash \tau_i : \Omega$. Instead of giving the whole set of rules for the calculi, we mention their salient features and illustrate them with representative instances of the rules. A detailed presentation is in [PDM89, Jac91a].

$\lambda \rightarrow$: Ω is the only kind. The judgements $\Theta \vdash \tau : \Omega$ and $\Theta \vdash \tau = \tau' : \Omega$, which introduce types and equate them respectively, correspond to a single-sorted algebraic theory. Ω is closed under finite products and exponentials, like Prop in 2.1.1. $\vdash \tau : \Omega$ are (given) closed types. For instance, we have the following derivable judgement

$$\frac{}{X; \Omega, Y : \Omega \vdash X \longrightarrow U : \Omega}$$

for type variables X and Y.

For every kind context Θ, judgements $\Theta|\Gamma \vdash t : \tau$ and $\Theta|\Gamma \vdash t = t' : \tau$ correspond to a simply typed λ-calculus, with the type variables declared in Θ among the types, in addition to the closed types, *e.g.*

$$\frac{X : \Omega|y : X, x : X \vdash x : X}{X : \Omega|y : X \vdash \lambda x : X.x : X}$$

There is a substitution of types for type variables in both types and term, *i.e.*

$$\frac{\Theta \vdash \tau : \Omega \quad X : \Omega|x : X \vdash t : \tau'}{\Theta|x : \tau \vdash t[X := \tau] : \tau'[X := \tau]}$$

A term $X : \Omega|\Gamma \vdash t : \tau$ is *polymorphic*, since it can instantiated at every type $\vdash \tau : \Omega$.

$\lambda 2$: Also known as system F [Gir86]. It extends $\lambda \rightarrow$ by allowing quantification on types:

$$\frac{\Theta, X : \Omega \vdash \tau : \Omega}{\Omega \vdash \Pi X : \Omega.\tau : \Omega}$$

At the term level, there is type abstraction:

$$\frac{\Theta, X : \Omega|\Gamma \vdash t : \tau}{\Theta|\Gamma \vdash \Lambda X : \Omega.t : \Pi X : \Omega.\tau}$$

where it is implicit that X does not occur free in the types of Γ. The term $\Lambda X : \Omega.t$ is explicitly polymorphic: it can be applied to (instantiated at) different types, *e.g.*

$$\frac{\Theta|\Gamma \vdash \Lambda X : \Omega.t : \Pi X : \Omega.X \rightarrow X \qquad \Theta \vdash \tau : \Omega}{\Theta|\Gamma \vdash ((\Lambda X : \Omega.t)(\tau \longrightarrow \tau))((\Lambda X : \Omega.t)\tau) : \tau \rightarrow \tau}$$

There are β and η rules for type abstraction and application.

$\lambda\omega$: Extends $\lambda 2$ by closing Kind under finite products and exponentials. So, kinds and kind terms $\Theta \vdash \tau : \kappa$ (which include the types) form a simply typed λ-calculus, *e.g.*

$$\frac{\Theta, X : \kappa \vdash \tau : \kappa'}{\Theta \vdash \lambda X : \kappa.\tau : \kappa \longrightarrow \kappa'}$$

Now, Ω is closed under quantification over all kinds:

$$\frac{\Theta, X : \kappa \vdash \tau : \Omega}{\Theta \vdash \Pi X : \kappa.\tau : \Omega}$$

and there is kind abstraction and application for terms, with β, η rules.

See [PDM89] for programming examples in the above calculi. These calculi are also logical systems: just as simply typed λ-calculus is the proof language of intuitionistic propositional logic, $\lambda 2$ and $\lambda\omega$ are the proof languages of second and higher-order intuitionistic propositional logic [Gir86]. Ω is then the kind of *proposition*, closed under impredicative quantification.

Categorically, the above calculi are interpreted in fibrations with structure. We follow [Jac91a, §3.3.2]. Kinds κ correspond to objects $[\![\kappa]\!]$ in the base category of a fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$. $\mathbb{B}$ has a distinguished object Ω, which interprets the kind Ω, and has finite products. Types and terms in context $\Theta = [X_1 : \kappa_1, \ldots, X_n : \kappa_n]$ correspond to objects and morphisms respectively of a cartesian closed category $\mathbb{E}_{[\![\Theta]\!]}$. This ccc is the fibre of p over the object $[\![\Theta]\!] = [\![\kappa_1]\!] \times \cdots \times [\![\kappa_n]\!]$. The type $X : \Omega \vdash X : \Omega$ is *generic*, in the sense that every type $\Theta \vdash \tau : \Omega$ is obtained from it by substitution. Hence $X : \Omega \vdash X : \Omega$ must be an object G in $\mathbb{E}_\Omega$, which is a generic object for

p, as in Definition 1.4.10. Thus, every type $\Theta \vdash \tau : \Omega$ determines a 'classifying' morphism $\lceil T \rceil : [\![\Theta]\!] \to \Omega$ in $\mathbb{B}$ such that $[\![T]\!] \simeq \lceil T \rceil^*(G)$ in $\mathbb{E}_{[\![\Theta]\!]}$. The reindexing functor $\lceil T \rceil : \mathbb{E}_\Omega \to \mathbb{E}_{[\![\Theta]\!]}$ performs type substitution in the types $X : \Omega \vdash \tau : \Omega$ and terms $X : \Omega | \Gamma t : \tau$, which have a type variable X. So, reindexing functors preserve cartesian closed structure. Type quantification $\Theta, X : \kappa \vdash \tau : \Omega \mapsto \Theta \vdash \Pi X : \kappa.\tau : \Omega$ is interpreted by a functor $\Pi : \mathbb{E}_{[\![\Theta]\!] \times [\![\kappa]\!]} \to \mathbb{E}_{[\![\Theta]\!]}$ such that $\pi^*_{[\![\Theta]\!],[\![\kappa]\!]} \dashv \Pi$. Type abstraction and application are then interpreted using the hom-set isomorphisms of this adjunction, analogously to exponentials in simply typed λ-calculus. For a proper interaction of quantification and substitution, the functors Π must satisfy the Beck-Chevalley condition.

2.1.3. Definition.

(i) A $\lambda \to$ fibration is a fibred-ccc $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ with a generic object G and $\mathbb{B}$ has finite products.

(ii) A $\lambda 2$-fibration is a $\lambda \to$-fibration with $Cons_\Omega$-products, where $\Omega = pG$.

(iii) A $\lambda\omega$-fibration is a $\lambda \to$-fibration, with $\mathbb{B}$ cartesian closed and which has $Cons_{\mathbb{B}}$-products.

2.1.4. Example. Let $\mathbb{C}$ be a small cartesian closed category.

- $f(\mathbb{C}) : Fam(\mathbb{C}) \to \mathcal{S}et$ is a $\lambda \to$-fibration. The cartesian closed structure in every fibre is given pointwise. Closed types correspond to objects of $(\mathit{Fam}(\mathbb{C}))_{\{*\}} \cong \mathbb{C}$. $\Omega = |\mathbb{C}|$ and $T = \{X\}_{X \in |\mathbb{C}|}$.

- If $\mathbb{C}$ is complete, $f(\mathbb{C})$ is a $\lambda\omega$-fibration, see Example 1.4.6.(i). By an argument of Freyd [Mac71, §V.2, Proposition 3] $\mathbb{C}$ must be a preorder. Hence every 'type' has at most one element. Remarkably, there are internal complete categories in realisability toposes which are not preorders [Hyl89, Pho92].

2.2 Logical predicates over applicative structures

The material in this section is from [Mit90]. Applicative structures – satisfying some conditions – provide a general notion of set-theoretic model for a simply typed λ-calculus. Let Σ be an algebraic signature providing basic types and terms for simply typed λ-calculus.

2.2.1. Definition. A *typed applicative structure* $\mathcal{A}$ for signature Σ is a tuple

$$\langle \{A_\sigma\}, \{\mathit{App}_{\sigma,\tau}\}, \mathit{Const} \rangle$$

of families of sets and functions indexed by type expressions σ, τ over the type constants from Σ, such that

- A_σ is a set.

- $\mathit{App}_{\sigma,\tau}$ is a set-theoretic map $\mathit{App}_{\sigma,\tau} : A_{\sigma \longrightarrow \tau} \to (A_\sigma \Rightarrow A_\tau)$.

- *Const* is a mapping from term constants of Σ to elements of the appropriate A_σ's, i.e. $Const(c) \in A_\sigma$ for every $c : \sigma$ in Σ.

If we want to consider $\times$-types, we may add an explicit interpretation for them, or simply assume they are interpreted as the cartesian product of the carriers of the corresponding types. An applicative structure is *extensional* if it satisfies the condition

$$\forall f, g \in A_{\sigma \longrightarrow \tau}.(\forall d \in A_\sigma.\ App_{\sigma,\tau} f d = App_{\sigma,\tau} g d) \Longrightarrow f = g$$

An *environment model* is an extensional applicative structure which can interpret all the terms of λ-calculus over Σ according to the obvious meaning function defined by structural induction on terms. Specifically, given a context Γ — a finite mapping of variables to type expressions over Σ — an environment ρ for it assigns to every $x : \sigma$ in Γ an element of A_σ. Define

$$\rho \models \Gamma \triangleq \forall x : \sigma \in \Gamma. \rho(x) \in A_\sigma$$

If $\rho \models \Gamma, \mathcal{A}[\![\Gamma \vdash t : \sigma]\!]\rho$ denotes the $\mathcal{A}$-value, in A_σ, of the term t of type σ in environment ρ. The interpretation of a lambda abstraction

$$[\![\Gamma \vdash \lambda x : \sigma.t : \sigma \longrightarrow \tau]\!]\rho$$

is the unique $f \in A_{\sigma \longrightarrow \tau}$ such that

$$\forall a \in A_\sigma.\ App_{\sigma,\tau} f a = [\![\Gamma, x : \sigma \vdash t : \tau]\!]\rho[a/x]$$

where $\rho[a/x]$ is the environment ρ extended by the assignment $x \mapsto a$. The environment model condition requires the existence of such f, whose uniqueness is guaranteed by extensionality. When $A_{\sigma \longrightarrow \tau} \cong A_\sigma \Rightarrow A_\tau$ we refer to the model as a *full type hierarchy*; the environment model condition is satisfied in this case.

2.2.2. Remark. There is a relation between environment models for simply typed λ-calculus and cartesian closed categories. For simplicity of presentation, we identify a type σ with the object which interprets it in a ccc, as in §2.1.1

- A ccc $\mathbb{C}$ gives rise to an applicative structure $\mathcal{A} : A_\sigma = \mathbb{C}(1, \sigma)$, $App_{\sigma,\tau} = \mathbb{C}(1, ev_{\sigma,\tau} \circ \langle \_, \_ \rangle) : \mathbb{C}(1, \sigma \Rightarrow \tau) \times \mathbb{C}(1, \sigma) \to \mathbb{C}(1, \tau)$, with the interpretations of constants as given for $\mathbb{C}$, *e.g.* a constant $\mathbb{C} : \sigma$ is interpreted in $\mathbb{C}$ as a morphism $c : 1 \to \sigma \in A_\sigma$. This structure is extensional when $\mathbb{C}(1, \_) : \mathbb{C} \to \mathcal{S}et$ is faithful. The environment model condition is satisfied because, given an environment ρ, the interpretation $\mathcal{A}[\![\Gamma \vdash t : \sigma]\!]\rho$ is obtained from that in $\mathbb{C}$ as $[\![\Gamma \vdash t : \sigma]\!] \circ \rho_\Gamma$. Here, for $\Gamma = [x_1 : \tau_1, \ldots, x_n : \tau_n]$, $\rho_\Gamma \in \mathbb{C}(1, \tau_1 \times \ldots \tau_n)$ is obtained by tupling the $\rho(x_i)$.

- An environment model $\mathcal{A}$, which interprets $\times$-types, generates a cartesian closed category $\mathbb{C}$ as follows: its objects are the types σ and

$$\begin{array}{c}\mathbb{C}(\sigma, \tau) = \{f : A_\sigma \to A_\tau \in \mathcal{S}et | \exists t.x : \sigma \vdash t : \tau \text{ and} \\ f(a) = [\![x : \sigma \vdash t : \tau]\!][a/x]\}\end{array}$$

  That is, we consider only those functions definable by terms of the calculus. The cartesian closed structure is easily obtained using the operations of the calculus on the terms defining the morphisms. The restriction to definable functions is not necessary for full type hierarchies, which are sub-ccc's of $\mathcal{S}et$.

Full details of the above relationship between the two notions of models for simply typed λ-calculus and generalisations to weaker calculi are given in [Mar92]. See also [Jac91b], where this relationship is extended to second-order λ-calculus.

We now recall the definition of a logical predicate over an applicative structure, as given in [Mit90].

2.2.3. Definition. Let $\mathcal{A} = \langle \{A_\sigma\}, \{App_{\sigma,\tau}\}, Const \rangle$ be an applicative structure for signature Σ. A *logical predicate* $\mathcal{P} = \{P_\sigma\}$ over $\mathcal{A}$ is a family of predicates indexed by type expressions over Σ such that

- $P_\sigma \subseteq A_\sigma$
- for all $f \in A_{\sigma \longrightarrow \tau} P_{\sigma \longrightarrow \tau}(f)$ iff $\forall x \in A_\sigma.\ P_\sigma(x) \Longrightarrow P_\tau(App_{\sigma,\tau} f x)$
- $P_\sigma(Const(c))$ for every constant $c : \sigma$ in Σ.

If we consider $\times$-types, we should add the following condition: for all $z \in A_{\sigma\times\tau}$,

$$P_{\sigma\times\tau}(z) \text{ iff } P_\sigma(\pi_{\sigma,\tau}(z)) \wedge P_\tau(\pi'_{\sigma,\tau}(z))$$

where $\pi_{\sigma,\tau} : A_{\sigma\times\tau} \to A_\sigma, \pi'_{\sigma,\tau} : A_{\sigma\times\tau} \to A$ denote the corresponding projections for the $\times$-type.

The notion of n-ary logical relation over n applicative structures corresponds then to a logical predicate over the product of the structures, which is again an applicative structure with the evident componentwise operations and interpretations of constants.

For an $\mathcal{A}$-environment ρ and a logical predicate $\mathcal{P}$ over $\mathcal{A}$, we define

$$\mathcal{P}(\rho) \triangleq \forall x : \sigma \in dom(rho).\ P_\sigma(\rho(x))$$

On the presence of $\times$-types, if $dom(\rho) = x_1 : \tau_1, \ldots, x_n : \tau_n, \mathcal{P}(\rho)$ amounts to $P_{\tau_1 \times \cdots \times \tau_n}(\rho(x_1), \ldots, \rho(x_n))$.

2.2.4. Lemma *(Basic Lemma, for models). Let $\mathcal{A}$ be an environment model for Σ and $\mathcal{P}$ be a logical predicate over $\mathcal{A}$. For all $\mathcal{A}$-environments ρ and Σ-typing contexts Γ, if $\rho \models \Gamma$ then*

$$\mathcal{P}(\rho) \Longrightarrow P_\sigma(\mathcal{A}[\![\Gamma \vdash t : \sigma]\!]\rho)$$

for every term $\Gamma \vdash t : \sigma$.

For a closed term $\vdash t : \sigma$, the above lemma implies that $P_\sigma(t)$ holds. The Basic Lemma is the fundamental property of logical predicates, which makes them a useful technical tool in proving properties about terms of simply typed λ-calculus. Given a property $\mathcal{Q}$ on terms, *e.g.* $\mathcal{Q}(t) = t$ is strongly normalising, if we want to prove that it holds for every term, we just have to find suitable logical predicate $\mathcal{P}$ which entails $\mathcal{Q}$; the desired result follows then from the Basic Lemma.

This situation is similar to the usual method of proving properties about the natural numbers by induction, where we must find a suitable inductive property which entails the desired one. For a precise analogy between logical predicates and induction, see §4.2 and §4.5.

Several applications of logical predicates are in [Mit90], including the following ones:

- extensional collapse of an applicative structure by a logical partial equivalence relation, *i.e.* symmetric and transitive on every type, which yields an extensional applicative structure.
- proofs of normalisation and confluence properties for λ-calculus,
- representation independence results for simply typed programming languages, stating that programs do not depend on the way data types are represented but only on the behaviour of data types with respect to the operations provided. This is formulated by requiring that if two interpretations $\mathcal{A}$ and $\mathcal{B}$ of the language are related in a 'certain way', then the meanings $\mathcal{A}[\![\vdash p : \sigma]\!]$ and $\mathcal{B}[\![\vdash p : \sigma]\!]$ of a program, *i.e.* a closed term, $\vdash p : \sigma$ are also related in the same 'certain way'. The precise way in which the interpretations must be related is via logical relations, which guarantee the desired property by the Basic Lemma.

2.3 Reflective and coreflective cartesian closed categories

We conclude our prelimary material with two propositions which allow to infer cartesian closed structure for reflective and coreflective categories. We will apply these propositions in §4.3 to infer the cartesian closure of some categories in the context of logical predicates.

2.3.1. PROPOSITION. *Given a coreflection of categories with finite products*

$$\begin{tikzcd} \mathbb{C} \arrow[r, "J"', shift right] \arrow[r, phantom, "\top"] & \mathbb{D} \arrow[l, "G"', shift right] \end{tikzcd}$$

where J is a full and faithful finite-product preserving functor, if $\mathbb{D}$ is cartesian closed, so is $\mathbb{C}$.

Proof. For $X, Y \in |\mathbb{C}|$, let

$$X \Rightarrow^{\mathbb{C}} Y \triangleq G(JX \Rightarrow^{\mathbb{D}} JY)$$

where we use superscripts to differentiate between exponentials in $\mathbb{C}$ and $\mathbb{D}$. Then,

$$\begin{array}{rcl} \mathbb{C}(Z, X \Rightarrow^{\mathbb{C}} Y) & \cong & \mathbb{C}(Z, G(JX \Rightarrow^{\mathbb{D}} JY)) \\ & \cong & \mathbb{D}(JZ, JX \Rightarrow^{\mathbb{D}} JY) \\ & \cong & \mathbb{D}(JZ \times^{\mathbb{D}} JX, JY) \\ & \cong & \mathbb{D}(J(Z \times^{\mathbb{C}} X), JY) \\ & & \quad \text{since } J \text{ preserves products} \\ & \cong & \mathbb{C}(Z \times^{\mathbb{C}} X, Y) \\ & & \quad \text{since } J \text{ is full and faithful} \end{array}$$

$\square$

2.3.2. PROPOSITION. *Let*

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, bend right, "J"'] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend right, "L"']
\end{tikzcd}
$$

be a reflection, i.e. J is full and faithful, with $(\Rightarrow^{\mathbb{D}} \circ J \times J^{op})$ a natural isomorphism. If $\mathbb{D}$ is cartesian closed, so is $\mathbb{C}$.

Proof. Let $X \Rightarrow^{\mathbb{C}} Y \triangleq L(JX \Rightarrow^{\mathbb{D}} JY)$. For a brief argument, observe

$$
\begin{aligned}
J(X \Rightarrow^{\mathbb{C}} Y) &= JL(JX \Rightarrow^{\mathbb{D}} JY) \\
&\cong JX \Rightarrow^{\mathbb{D}} JY
\end{aligned}
$$

where the isomorphism is obtained using $\eta(\Rightarrow^{\mathbb{D}} \circ J \times J^{op})_{(X,Y)}$. This is the hypothesis of Lemma 4 of [Ehr89], which yields the desired conclusion. A direct calculation is just as simple and we leave it to the reader. $\square$

2.3.3. REMARK. The hypothesis of Proposition 3.3.11 means that the reflection does not affect exponential objects of the form $JX \Rightarrow^{\mathbb{D}} JY$. Indeed we could extend the above proposition to show that $\mathbb{C}$ is an exponential ideal of $\mathbb{D}$, meaning that for objects $X, Y \in |D|$ if Y is in $\mathbb{C}$, so is $X \Rightarrow^{\mathbb{D}} Y$. This applies to categories of sets with structure, where the structure on exponentials is given pointwise.

Chapter 3

Fibred adjunctions and change of base

In this chapter we examine the relationship between change-of-base and fibred adjunctions. The main result, Theorem 3.2.3, shows that by performing change-of-base along a left adjoint functor we can factorise a fibred adjunction into a standard adjunction in $\mathcal{C}at$ and a vertical fibred adjunction. Such a factorisation has two immediate important corollaries: Corollary 3.3.6, which characterises fibred limits for a fibration in terms of limits for its total and base categories and Corollary 3.3.11, which proves the Cartesian closed property of the total category of a fibred-ccc with products. This second corollary will be applied in §4.2 to give a category-theoretic account of logical predicates for the simply typed λ-calculus. A further important application of Theorem 3.2.3 occurs in §5, in the construction of the Kieisli fibration of a comonad in $\mathcal{F}ib$.

In order to prove the abovementioned result we present in the first section several properties of change-of-base, which essentially rephrase the elementary formulation of cartesian morphisms in terms of 2-cells. These properties allow us to deal with adjunctions in an algebraic way, using the 2-categorical definition 1.1.3. Not only does this give an elegant proof but it also shows that the argument can be carried over to fibred 2-categories. See §7 for further considerations on this topic.

3.1 Change-of-base and 2-categorical structure

In this section we collect together some 2-categorical aspects of the change-of-base construction. First, there is a change-of-base 2-functor induced by a [Jac91a, Lemma 1.1.7].

3.1.1. LEMMA. *Every functor $K : \mathbb{B} \to \mathbb{A}$ induces a change-of-base 2-functor $K^* : \mathcal{F}ib(\mathbb{A}) \to \mathcal{F}ib(\mathbb{B})$, which preserves finite products.*

$$\begin{tikzcd}
K^*(\mathbf{E}) \arrow[rr, shift left, "K^*(F)"] \arrow[rr, shift right, "K^*(G)"'] \arrow[rr, phantom, "\Downarrow K^*(\sigma)"] \arrow[dr, "K^*(p)"] \arrow[ddd] & & K^*(\mathbf{D}) \arrow[dl, "K^*(q)"'] \arrow[ddd] \\
& \mathbb{B} \arrow[d, "K"] & \\
& \mathbb{A} & \\
\mathbf{E} \arrow[ur, "p"] \arrow[rr, shift left, "F"] \arrow[rr, shift right, "G"'] \arrow[rr, phantom, "\Downarrow \sigma"] & & \mathbf{D} \arrow[ul, "q"']
\end{tikzcd}$$

This 2-functor restricts to $K^* : \mathcal{F}ib_{sp}(\mathbb{A}) \to \mathcal{F}ib_{sp}(\mathbb{B})$.

The following lemma expresses the cartesian-lifting property of fibrations in terms of 2-cells (*i.e.* natural transformations). The first item is [Jac91a, Lemma 1.1.8].

3.1.2. LEMMA.

(i) Given $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{A}\end{smallmatrix}$ *and a natural transformation* $(\sigma : K \dashrightarrow L) : \mathbb{B} \to \mathbb{A}$. *There is* $\mathbb{B}$*-fibred 1-cell* $\langle\sigma\rangle_q : L^*(q) \to K^*(q)$ *and a 2-cell* $\sigma'_q : q^{ast}(K) \circ \langle\sigma\rangle_q \Rightarrow q^*(L)$,

$$\begin{tikzcd}
L^*(\mathbf{D}) \arrow[rr, "q^*(L)"] \arrow[dr, "\langle\sigma\rangle_q"] \arrow[ddr, "L^*(q)"'] & \arrow[d, phantom, "\Uparrow \sigma'_q"] & \mathbf{D} \arrow[dd, "q"] \\
& K^*(\mathbf{D}) \arrow[ur, "q^*(K)"'] \arrow[dl, "K^*(q)"'] & \\
\mathbf{B} \arrow[rr, shift left, "L"] \arrow[rr, shift right, "K"'] \arrow[rr, phantom, "\Uparrow \sigma"] & & \mathbf{A}
\end{tikzcd}$$

such that $(\sigma'_q, \sigma) : (q^*(K) \circ \langle\sigma\rangle, K) \Rightarrow (q^*(L), L) : L^*(q) \to q$ *is a fibred 2-cell and* σ'_q *has cartesian components.*

(ii) Given $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ and $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{A}\end{smallmatrix}$, *a natural transformation* $\sigma : K \dashrightarrow L : \mathbb{B} \to \mathbb{A}$ *and two 1-cells* $(\tilde{K}, K), (\tilde{L}, L) : p \to q$ *in* $\mathcal{C}at^{\to}$, *change-of-base induces a one-to-one correspondence between vertical 2-cells (that is, 2-cells in* $\mathcal{C}at/\mathbb{B}$*)*

$$\hat{\sigma} : \hat{K} \Rightarrow \langle\sigma\rangle_q \circ \hat{L} : p \to K^*(q)$$

and 2-cells $(\tilde{\sigma}, \sigma) : (\tilde{K}, K) \Rightarrow (\tilde{L}, L) : p \to q$.

$$
\begin{tikzcd}
& K^*(q) \arrow[dr, "q^*(K)"] & \\
p \arrow[ur, "\hat{K}"] \arrow[dr, "\hat{L}"'] \arrow[r, phantom, "\Downarrow\hat{\sigma}"] & {} \arrow[r, phantom, "\Downarrow\sigma'_q"] & q \\
& L^*(q) \arrow[uu, "\langle\sigma\rangle_q"] \arrow[ur, "q^*(L)"'] &
\end{tikzcd}
\quad
\Big(= \quad
\begin{tikzcd}
p \arrow[r, shift left=2, "\tilde{K}"] \arrow[r, shift right=2, "\tilde{L}"'] \arrow[r, phantom, "\downarrow\tilde{\sigma}"] & q
\end{tikzcd}\Big)
$$

Proof. During the proof we omit the subscripts $_q$ to simplify notation.

(i) For $X \in |L^*(\mathbb{D})|$, σ'_X is the unique morphism f such that $q^*(L)f = \overline{\sigma_{L^*(q)X}}^q(q^*(L)X)$ and $\langle\sigma\rangle_q X$ is the unique object Y of $K^*(\mathbb{D})$ such that $q^*(K)Y = \sigma^{*q}_{L^*(q)X}(q^*(L)X)$; the morphism part of $\langle\sigma\rangle_q$ is similarly determined by the universal property of cartesian morphisms. $\langle\sigma\rangle_p$ preserves Cartesian morphisms by Proposition 1.2.3.

(ii) By (i), we get $\sigma' : q^*(K) \circ \langle\sigma\rangle \Rightarrow q^*(L)$. So, given $\hat{\sigma} : \hat{K} \Rightarrow \langle\sigma\rangle \circ \hat{L}$, we get $\tilde{\sigma} : \tilde{K} \Rightarrow \tilde{L}$ over σ by composition:

$$\tilde{\sigma} = \sigma'\hat{L} \circ q^*(K)\hat{\sigma}$$

Conversely, given $\tilde{\sigma} : \tilde{K} \Rightarrow \tilde{L}$, we get $\hat{\sigma} : \hat{K} \Rightarrow \langle\sigma\rangle \circ \hat{L}$ as follows: σ' is cartesian over σ and therefore, for $X \in |\mathbb{E}|$ we have

$$
\begin{tikzcd}
q^*(K)\hat{K}X \arrow[dr, "\exists!\phi_{\tilde{\sigma}_X}"'] \arrow[drr, "\tilde{\sigma}_X"] & & & \\
& q^*(K)\langle\sigma\rangle\hat{L}X \arrow[r, "\sigma'_{\hat{L}X}"'] & q^*(L)\hat{L}X & {} \arrow[ddd, dotted, "q"] \\
& & & \\
KpX \arrow[dr, "1_{KpX}"'] \arrow[drr, "\sigma_{pX}"] & & & \\
& KpX \arrow[r, "\sigma_{pX}"'] & LpX & {}
\end{tikzcd}
$$

$\hat{\sigma}_X : \hat{K}X \to \langle\sigma\rangle\hat{L}X$ is the unique morphism in $K^*(\mathbb{D})$ such that

$$
\begin{aligned}
q^*(K)\hat{\sigma}_X &= \phi_{\tilde{\sigma}X} \\
K^*(q)\hat{\sigma}_X &= 1_{pX}
\end{aligned}
$$

obtained using the 2-dimensional property of pullbacks in $\mathcal{C}at$, see §1.1. Naturality of $\hat{\sigma}$ follows readily from the universality of Cartesian liftings. The constructions of $\hat{\sigma}$ and $\tilde{\sigma}$ are mutually inverse. □

We say that $\hat{\sigma}$ above is obtained by factoring $\tilde{\sigma}$ through σ. We call the 2-cell (σ', σ) in (i) a *cartesian fibred 2-cell*. The notation $\hat{\sigma}, \langle\sigma\rangle_q$ and σ'_q will be used in q the remaining of the thesis.

3.1.3. Remark. Statement (i) above asserts the existence of a cartesian lifting for natural transformations. (ii) asserts its universal property.

Using Lemma 3.1.2 we obtain the following condition for a functor $F : p \to q$ in $\mathcal{C}at/\mathbb{B}, p, q$ fibrations over $\mathbb{B}$, to be a $\mathbb{B}$-fibred 1-cell. First we need an auxiliary definition.

3.1.4. Definition. Given fibrations $p : \mathbb{E} \to \mathbb{B}$ and $q : \mathbb{D} \to \mathbb{B}, F : p \to q$ and a natural transformation $\sigma : K \dashrightarrow H : \mathbb{A} \to \mathbb{B}$ let $\langle\sigma\rangle_p : H^*(p) \to K^*(p)$ and $\sigma' : p^*(K) \circ \langle\sigma\rangle_p \Rightarrow p^*(H)$ be the fibred 1-cell and 2-cell obtained by applying Lemma 3.1.2.(i) to σ and p. Similarly, let $\langle\sigma\rangle_q : H^*(q) \to K^*(q)$ and $\sigma'_q : q^*(K) \circ \langle\sigma\rangle_q \Rightarrow q^*(H)$ be the corresponding ones for σ with q. Then, considering

$$F\sigma'_p : q^*(K) \circ K^*(F) \circ \langle\sigma\rangle_p \Rightarrow F \circ p^*(H)(= q^*(H) \circ H^*(F))$$

we have a 2-cell

$$(F\sigma'_p, \sigma) : (q^*(K) \circ K^*(F) \circ \langle\sigma\rangle_p K, K) \Rightarrow (q^*(H) \circ H^*(F), H)$$

in $\mathcal{C}at^{\to}$, which induces by Lemma 3.1.2.(ii) a vertical 1-cell

$$\phi^F_\sigma : K^*(F) \circ \langle\sigma\rangle_p \Rightarrow \langle\sigma\rangle_q \circ H^*(F)$$

ϕ^F_σ is the *canonical comparison 2-cell*.

From the proof of Lemma 3.1.2.(ii), the component at an object $(I, X) \in |\mathbb{A} \underset{K,q}{\times} \mathbb{E}|$ of the canonical comparison 2-cell ϕ^F_σ is the canonical morphism $(I, F\sigma_I^{*p}(X)) \to (I, \sigma^{*q}(FX))$ in $\mathbb{A} \underset{f,g}{\times} \mathbb{E}$, induced by the cartesian morphism $\overline{(\sigma_I)}^q(FX) : \sigma^{*q}(FX) \to FX$. Hence, F preserves cartesian morphisms precisely when every component of ϕ^F_σ is an isomorphism, as stated in the following proposition.

3.1.5. Proposition. *Given* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$, $\begin{smallmatrix}\mathbb{D}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ $F : p \to q$ *in* $\mathcal{C}at/\mathbb{B}$, F *is a* $\mathbb{B}$*-fibred 1-cell (i.e. preserves cartesian morphisms) iff for every natural transformation* $\sigma : K \dashrightarrow H : \mathbb{A} \to \mathbb{B}$ *(for arbitrary* $\mathbb{A}$*), the canonical comparison 2-cell* ϕ^F_σ *is an isomorphism. Further,* F *preserves cleavages (strictly) iff* ϕ^F_σ *is the identity.*

3.1.1 Algebra of fibred 2-cells

In this section we state some equational laws for fibred 2-cells. They follow from the universal property of Cartesian 2-cells, as stated in Lemma 3.1.2. These laws will be used in §3.2 to prove properties about adjunctions in $\mathcal{F}ib$ and in §5.4.2, where we present Kleisli fibrations for comonads in $\mathcal{F}ib$.

Given functors $K : \mathbb{A} \to \mathbb{B}$ and $H : \mathbb{B} \to \mathbb{C}$, we have by Lemma 3.1.1 2-functors $K^* \circ H^* : \mathcal{F}ib(\mathbb{C}) \to \mathcal{F}ib(\mathbb{A})$ and $(HK)^* : \mathcal{F}ib(\mathbb{C}) \to \mathcal{F}ib(\mathbb{A})$, given by pullbacks. Since the composite of two pullbacks squares is again a pullback, by Corollary 1.2.4, there is a 2-natural isomorphism $J_{K,H} : K^* \circ H^* \Rightarrow (HK)^*$. Further, we assume that $1^*_{\mathbb{A}} = 1_{\mathcal{F}ib(\mathbb{A})}$ We thus have a normalised cleavage for the fibration $cod : \mathcal{F}ib \to \mathcal{C}at$. These J's satisfy coherence conditions as for the δ's in Definition 1.3.1.

The following two propositions summarise the algebraic laws concerning fibred 2-cells. They essentially give an 'external' formulation of the elementary properties of cartesian and vertical morphisms of a fibred category. They show a 2-dimensional aspect of the fibration $cod : \mathcal{F}ib \to \mathcal{C}at$. They can be verified using the elementary definition of the natural transformations involved and universality of Cartesian morphisms. An equivalent and simpler way to prove them is by 'pasting' of 2-cells and Lemma 3.1.2.(ii). We prove one of them, Lemma 3.1.7, for illustration.

Since the objects of $\mathcal{F}ib$ are fibrations, their Cartesian lifting property determines vertical *comparison* 2-cells, as given by Lemma 3.1.2.(ii) . In detail, for $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$,

- given a functor $F : \mathbb{A} \to \mathbb{B}$, p induces a 2-cell $\gamma^p_F : 1_{F^*(p)} \Rightarrow \langle 1_F\rangle_p$, whose component at an object $\langle I, X\rangle \in |\mathbb{A} \underset{F,P}{\times} \mathbb{E}|$ is $\langle 1_I, \gamma_I : X \to 1^*_I(X)\rangle$, induced by $\overline{1_I}(X)$;
- given 2-cells $\mathbb{A} \overset{H}{\underset{K\Downarrow\beta}{\overset{J\Downarrow\alpha}{\rightrightarrows}}} \mathbb{B}$, p induces a 2-cell $\delta^p_{\alpha,\beta} : \langle\alpha\rangle_p \circ \langle\beta\rangle_p \Rightarrow \langle\beta\circ\alpha\rangle_p$, whose component at $\langle I, X\rangle \in |\mathbb{A} \underset{K,p}{\times} \mathbb{E}|$ is $\langle 1_I, \delta_{\alpha_I,\beta_I} : \alpha^*_I(\beta^*_I(X)) \to (\beta_I \circ \alpha_I)^*(X)\langle$ induced by $\overline{\beta_I \circ \alpha_I}(X)$

γ's and δ's are uniquely characterised by the following properties:

- $(1_F)'_p \circ p^*(F)\gamma^p_F = 1_{p^*(F)}$
- $(\beta\circ\alpha)'_p \circ p^*(H)\delta^p_{\alpha,\beta} = \beta'_p \circ \alpha'_p\langle\beta\rangle_p$

These 2-cells satisfy the coherence conditions of Definition 1.3.1. When the γ's are identities, we have a normalised cleavage for p. If additionally the δ's are identities, we a have a splitting. The comparison 2-cells are preserved by change-of-base, as stated in the following lemma.

3.1.6. Lemma. *Given* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *and* $\mathbb{C} \xrightarrow{L} \mathbb{A} \underset{K \Downarrow \beta}{\overset{H}{\underset{J \Downarrow \alpha}{\rightrightarrows}}} \mathbb{B}$,

- $J_{K,F}(p)L^*(\gamma^p_H) = \gamma^p_{HL} J_{L,H}(p)$
- $J_{L,H}(p)L^*(\delta^p_{\alpha,\beta}) = \delta^p_{\alpha L,\beta L} J_{L,K}(p)$

The comparison 2-cells are also preserved by fibred functors, as follows:

3.1.7. Lemma. *Given a* $\mathbb{B}$*-fibred 1-cell* $F : p \to q$ *and 2-cells* $\mathbb{A} \underset{K \Downarrow \beta}{\overset{H}{\underset{J \Downarrow \alpha}{\rightrightarrows}}} \mathbb{B}$,

$$\begin{tikzcd}
H^*(p) \arrow[rr, "H^*(F)"] \arrow[d, shift right=3, "1_{H^*(p)}"'] \arrow[d, shift left=3, "\langle 1_H \rangle_p"] \arrow[d, phantom, "\overset{\gamma^p_H}{\Rightarrow}"] \arrow[drr, phantom, "\overset{\phi}{\Rightarrow}"] & & H^*(q) \arrow[d, "\langle 1_H \rangle_q"] \\
H^*(p) \arrow[rr, "H^*(F)"'] & & H^*(q)
\end{tikzcd}
=
\begin{tikzcd}
H^*(p) \arrow[rr, "H^*(F)"] \arrow[d, "1_{H^*(p)}"'] & & H^*(q) \arrow[d, shift right=3, "1_{H^*(q)}"'] \arrow[d, shift left=3, "\langle 1_H \rangle"] \arrow[d, phantom, "\overset{\gamma^q_H}{\Rightarrow}"] \\
H^*(p) \arrow[rr, "H^*(F)"'] & & H^*(q)
\end{tikzcd}$$

- *i.e.* $\phi^F_{1_H} \circ H^*(F)\gamma^p_H = \gamma^q_H H^*(F)$

$$\begin{tikzcd}
K^*(p) \arrow[r, "K^*(F)"] \arrow[d, "\langle\beta\rangle"'] \arrow[dd, bend left, "\langle\beta\circ\alpha\rangle"] \arrow[d, phantom, shift left=3, "\overset{\delta^p}{\Rightarrow}"] & K^*(q) \arrow[dd, "\langle\beta\circ\alpha\rangle"'] & & K^*(p) \arrow[r, "K^*(F)"] \arrow[d, "\langle\beta\rangle"'] \arrow[dr, phantom, "\overset{\phi_\beta}{\Rightarrow}"] & K^*(q) \arrow[d, "\langle\beta\rangle"'] \arrow[dd, bend left, "\langle\alpha\circ\beta\rangle"] \arrow[d, phantom, shift left=3, "\overset{\delta^q}{\Rightarrow}"] \\
J^*(p) \arrow[d, "\langle\alpha\rangle"'] \arrow[r, phantom, "\overset{\phi_{\beta\circ\alpha}}{\Rightarrow}"] & & = & J^*(p) \arrow[r, "J^*(F)"] \arrow[d, "\langle\alpha\rangle"'] \arrow[dr, phantom, "\overset{\phi_\alpha}{\Rightarrow}"] & J^*(q) \arrow[d, "\langle\alpha\rangle"'] \\
H^*(p) \arrow[r, "H^*(F)"] & H^*(q) & & H^*(p) \arrow[r, "H^*(F)"] & H^*(q)
\end{tikzcd}$$

- *i.e.* $\phi^F_{\beta\circ\alpha} \circ H^*(F)\delta^p_{\alpha,\beta} = \delta^q_{\alpha,\beta}K^*(F) \circ \langle\alpha\rangle_q\phi^F_\beta \circ \phi^F_\alpha\langle\beta\rangle_q$

Proof. We prove the second statement. We show both 2-cells are equal when composed with $(\beta\circ\alpha)'_q$:

$$\begin{aligned}
& (\beta\circ\alpha)'_q K^*(F) \circ q^*(H)(\phi^F_{\beta\circ\alpha} \circ H^*(F)\delta^p_{\alpha,\beta}) \\
= \;& (\beta\circ\alpha)'_p \circ q^*(H) \circ H^*(F)\delta^p_{\alpha,\beta}) \\
= \;& \beta'_p \circ \alpha'_p\langle\beta\rangle_p \\
= \;& \beta'_q K^*(F) \circ q^*(J)\phi^F_\beta \circ alpha'_q J^*(F)\langle\beta\rangle_p \circ q^*(H)\phi^F_\alpha\langle\beta\rangle_p \\
= \;& \beta'_q K^*(F) \circ \alpha'_p(J)\langle\beta\rangle_q K^*F \circ q^*(H)(\langle\alpha\rangle_q\phi^F_\beta \circ \phi^F_\alpha\langle\beta\rangle_p) \\
= \;& (\beta\circ\alpha)'_q \circ q^*(H)(\delta^q_{\alpha,\beta}K^*(F) \circ \langle\alpha\rangle_q\phi^F_\beta \circ \phi^F_\alpha\langle\beta\rangle_p)
\end{aligned}$$

using the defining properties of the δ and ϕ isomorphisms. The result follows by Lemma 3.1.2.(ii).

3.1.8. Lemma. *Consider the following data*

$$
\begin{tikzcd}
\tilde{\mathbf{A}} \arrow[r, shift left=2, "\tilde{H}"] \arrow[r, shift right=2, "\tilde{J}"'] \arrow[r, phantom, "\Downarrow\tilde{\alpha}"] \arrow[d, "q"'] & \tilde{\mathbf{B}} \arrow[r, shift left=2, "F"] \arrow[r, shift right=2, "G"'] \arrow[r, phantom, "\Downarrow\tilde{\gamma}"] \arrow[d, "p"'] & \tilde{\mathbf{C}} \arrow[dl, "p'"] \\
\mathbf{A} \arrow[r, shift left=2, "H"] \arrow[r, shift right=2, "J"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{B} &
\end{tikzcd}
$$

i.e. $\tilde{\gamma} : F \Rightarrow G : p \to p'$ *in* $\mathcal{F}ib(\mathbb{B})$ *and* $(\tilde{\alpha}, \alpha) : (\tilde{H}, H) \Rightarrow (\tilde{J}, J) : q \to p$ *in* $\mathcal{F}ib$. *Then*

$$
\begin{array}{ll}
(i) & \widehat{F\tilde{\alpha}} = \phi^F_\alpha \hat{J} \circ H^*(F)\hat{\alpha} \\
(ii) & \langle\alpha\rangle_{p'} \circ J^*(\tilde{\gamma}) \circ \phi^F_\alpha = \phi^G_\alpha \circ H^*(\tilde{\gamma}) \circ \langle\alpha\rangle_p
\end{array}
$$

3.1.9. Lemma. *Consider the following data.*

$$
\begin{tikzcd}
\tilde{\mathbf{C}} \arrow[r, "\tilde{L}"] \arrow[d, "r"'] & \tilde{\mathbf{A}} \arrow[r, shift left=4, "\tilde{H}"] \arrow[r, "\tilde{J}" near start] \arrow[r, shift right=4, "\tilde{K}"'] \arrow[r, phantom, shift left=2, "\Downarrow\tilde{\alpha}"] \arrow[r, phantom, shift right=2, "\Downarrow\tilde{\beta}"] \arrow[d, "q"'] & \tilde{\mathbf{B}} \arrow[r, "\tilde{M}"] \arrow[d, "p"] & \tilde{\mathbf{D}} \arrow[d, "s"] \\
\mathbf{C} \arrow[r, "L"'] & \mathbf{A} \arrow[r, shift left=4, "H"] \arrow[r, "J" near start] \arrow[r, shift right=4, "K"'] \arrow[r, phantom, shift left=2, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2, "\Downarrow\beta"] & \mathbf{B} \arrow[r, "M"'] & \mathbf{D}
\end{tikzcd}
$$

i.e. fibrations r, q, p, s, *fibred 1-cells* $(\tilde{L}, L) : r \to q, (\tilde{H}, H), (\tilde{J}, J), (\tilde{K}, K) : q \to p$ *and* $(\tilde{M}, M) : p \to s$ *and fibred 2-cells* $(\tilde{\alpha}, \alpha) : (\tilde{H}, H) \Rightarrow (\tilde{J}, J)$ *and* $(\tilde{\beta}, \beta) : (\tilde{J}, J) \Rightarrow (\tilde{K}, K)$.

$$\begin{array}{rl}
(i) & (\beta\circ\alpha)'_p = \beta'_p \circ \alpha'_p\langle\beta\rangle_p \circ p^*(H)(\delta^p_{\alpha,\beta})^{-1} \\
(ii) & \widehat{\tilde{\beta}\tilde{\alpha}} = \delta_{\alpha,\beta}\circ\langle\alpha\rangle\tilde{\beta}\circ\tilde{\alpha} \\
(iii) & L^*(\tilde{\alpha}) = \tilde{\alpha}\widehat{q^*(L)} \\
(iv) & \widehat{\tilde{\alpha}\tilde{L}} = L^*(\tilde{\alpha})\widehat{L} \\
(v) & \widehat{\tilde{M}\tilde{\alpha}} = \phi_\alpha\widehat{J}\circ H^*(\widehat{M})\tilde{\alpha}
\end{array}$$

3.1.10. LEMMA. *Given* $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{C}\end{array}$, $\begin{array}{c}\mathbb{E}'\\ \downarrow q\\ \mathbb{B}\end{array}$ *and* $\mathbb{D}\xrightarrow{L}\mathbb{A}\underset{J}{\overset{H}{\underset{\Downarrow\alpha}{\rightrightarrows}}}\mathbb{B}\xrightarrow{K}\mathbb{C}$

$$\begin{array}{rl}
(i) & \langle K\alpha\rangle_p = J_{H.K}(p)\langle\alpha\rangle_{K^*(p)}J^{-1}_{J,K}(p) \\
(ii) & (K\alpha)'_p = p^*(K)(\alpha)'_{K^*(p)}J^{-1}_{J,K}(p) \\
(iii) & \langle\alpha L\rangle_q = J_{L,H}(q)L^*(\langle\alpha\rangle_q)J^{-1}_{L,J}(q) \\
(iv) & (\alpha L)'_q = (\alpha)'_q(J^*(q))^*(L)J^{-1}_{L,J}(q)
\end{array}$$

3.2 Lifting and factorisation of adjunctions

We now have enough machinery to study the interaction between change-of-base and fibred adjunctions. The following lemma establishes one important aspect of the change-of-base 2-functors with respect to adjunctions between the base categories. We present two proofs for it to illustrate the difference between an intrinsic yet elementary reasoning, 'looking inside the categories' and a 2-categorical one, using the algebraic laws for fibred 2-cells in §3.1.1. The latter is more involved, but shows the argument in its natural context. In the proofs we will assume without loss of generality that the chosen cleavages are normalised.

3.2.1. LEMMA. *Given* $\begin{array}{c}\mathbb{E}\\ \downarrow q\\ \mathbb{B}\end{array}$ *and an adjunction* $F\dashv G:\mathbb{B}\to\mathbb{A}$ *via* η,ϵ, *change-of-base along F yields a cartesian fibred adjunction*

$$\begin{tikzcd}
F^*(\mathbb{E}) \arrow[r, shift left=1.5, "q^*(F)"] \arrow[r, phantom, "\perp"] \arrow[d, "F^*(q)"'] & \mathbb{E} \arrow[l, shift left=1.5, "\overline{G}"] \arrow[d, "q"] \\
\mathbb{A} \arrow[r, shift left=1.5, "F"] \arrow[r, phantom, "\perp"] & \mathbb{B} \arrow[l, shift left=1.5, "G"]
\end{tikzcd}$$

First Proof. Let X be an object of $\mathbb{E}_I$ We must show there is a cofree object for $\langle I, X\rangle$ with respect to F', *i.e*, terminal in the comma category $F' \downarrow \langle I, X\rangle$. It is $\overline{G}\langle I, X\rangle = \langle GI, \epsilon_I^*(X))\rangle \in |F^*(\mathbb{E})_{GI}|$ with counit component $\overline{\epsilon_I}(X) : \epsilon_I^*(X) \to X$ in $F^*(\mathbb{E})$. Here $\overline{\epsilon_I}$ is *any* cartesian morphism over ϵ_I with codomain X. As for universality, let $\langle J, Y\rangle \in |F^*(\mathbb{E})|$ and $f : Y \to X$ in $\mathbb{E}$. By universality of ϵ_I there exists a unique $f' : J \to GI$ such that $\epsilon_I \circ Ff' = qf$. $\overline{\epsilon_I}(X)$ is cartesian and hence there is a unique $h : Y \to \epsilon_I^*(X)$ with $\overline{\epsilon_I}(X) \circ h = f$ and thus a unique $\langle f', h\rangle : \langle J, Y\rangle \to \langle GI, \epsilon_I^*(X)\rangle$ with the required property, as shown in the diagram below

$$\begin{tikzcd}
\overline{G}(X) & & \epsilon_I^*(X) \arrow[r, "\overline{\epsilon_I}(X)"] & X \\
& {} \arrow[r, dotted, "F'"] & {} & \\
\langle J, Y\rangle \arrow[uu, dashed, "{\langle f', h\rangle}"] & & Y \arrow[uu, "h"] \arrow[uur, "f"'] &
\end{tikzcd}$$

over

$$\begin{tikzcd}
GI & & FGI \arrow[r, "\epsilon_I"] & I \\
& {} \arrow[r, dotted, "F"] & {} & \\
J \arrow[uu, dashed, "f''"] & & FJ \arrow[uu, "Ff'"] \arrow[uur, "qf"'] &
\end{tikzcd}$$

The construction given above is such that the counit of $q^*(F) \dashv overlineG$ is cartesian over that of $F \dashv G$, as required for a cartesian fibred adjunction. We could verify directly that the resulting functor $\overline{G}$ preserves cartesian morphisms, but this follows from Lemma 3.3.3.(ii) below. $\square$

Second Proof. We use the following abbreviations

$$q' = F^*(q), \quad F' = q^*(F), \quad G'' = (q')^*(G)$$

We have

$$\begin{tikzcd}
q \arrow[dr, "1_q"] \arrow[d, "\langle\epsilon\rangle_q"'] \arrow[dr, phantom, shift right=3ex, "\Uparrow \epsilon'_q"] & \\
(FG)^*(q) \arrow[r, "q^*(FG)"'] & q
\end{tikzcd}$$

The right adjoint $\overline{G}$ is $G'' \circ J^{-1}_{G,F}(q) \circ \langle\epsilon\rangle_q : q \to F^*(q)$, with counit $\overline{\epsilon} = \epsilon'_q : F'\overline{G} \Rightarrow 1_q$, since $F' \circ G'' \circ J^{-1}_{G,F}(q) = q^*(FG)$, by universality of pullbacks. The unit is $\overline{\eta} = (\eta'_q J^{-1}_{GF,F}(q)\langle\epsilon F\rangle_q)\delta^{-1}_{F_{\eta,\epsilon F}}$. The unit has the appropriate domain, that $\delta^{-1}_{F_{\eta,\epsilon F}}$ and codomain $(q')^*(GF)J^{-1}_{GF,F}(q)\langle\epsilon F\rangle_q = \overline{G}F'$ using Lemma

3.1.10.(iii) f and coherence conditions on J's. The triangular laws are obtained as follows:

- $\overline{\epsilon}F' \circ F'\overline{\eta} = 1_{F'}$ because $\overline{\epsilon}F' = (\epsilon F)'_q : q^*(FGF)\langle\epsilon F\rangle_q \Rightarrow 1_{F'}$, by Lemma 3.1.10.(iv) and $F'\overline{\eta} = (F\eta)'_q\langle\epsilon F\rangle_q \circ F'\delta^{-1}_{F\eta,\epsilon F}$ by Lemma 3.1.10.(ii). The result follows by Lemma 3.1.9.(i) and the adjunction laws for $F \dashv G$.

- To show $\overline{G}\overline{\epsilon} \circ \overline{\eta}\overline{G} = 1_{\overline{G}}$ it suffices to show $F'(\overline{G}\overline{\epsilon} \circ \overline{\eta}\overline{G} = 1_{F'\overline{G}}$ and $q'(\overline{G}\overline{\epsilon} \circ \overline{\eta}\overline{G} = 1_{G(FG)^*(q)}$ by the universal property of pullbacks. The latter is trivial. For the former

  $$F'\overline{G}\overline{\epsilon} = (FG\epsilon)'_q\langle\epsilon\rangle_q q^*(FGFG)(\delta^{-1}_{FG\epsilon,\epsilon} \circ \delta_{\epsilon FG,\epsilon})$$

  using Lemmas 3.1.8.(i) and 3.1.10.(iv), and coherence for J's,

  $$F'\overline{\eta}\overline{G} = (F\eta F)'_q\langle\epsilon FG\rangle_q\langle\epsilon\rangle_q \circ q^*(FG)(\langle F\eta G\rangle_q(\delta^{-1}_{\epsilon FG,\epsilon} \circ \delta_{FG\epsilon,\epsilon}) \circ \delta^{-1}_{FetaG,FG\epsilon})\langle\epsilon\rangle_q$$

  using Lemmas 3.1.10.(iv), 3.1.8.(ii) and 3.1.2.(ii). The result follows using the interchange law, cancelling opposite isomorphisms, and applying Lemma 3.1.9.(i) and the adjunction laws for $F \dashv G$.

  □

Theorem 3.2.3 below characterises fibred left adjoints in terms of vertical fibred ones or, more precisely, adjunctions in $\mathcal{F}ib$ in terms of adjunctions in $\mathcal{F}ib(\_)$ and $\mathcal{C}at$. For a concise statement, we introduce the following auxiliary definition

3.2.2. Definition. For a 2-category $\mathcal{K}$, let $\mathcal{K}_{Iadj}$ be the sub-2-category of

$\mathcal{K}$, with the same objects and 2-cells but with only those 1-cells $f : A \to B$ which have a right adjoint $f \dashv g$. Since the composite of two such 1-cells has a right adjoint, namely the composite of the given right adjoints, $\mathcal{K}_{ladj}$ is indeed a sub-2-category. Similarly, we have $\mathcal{K}_{radj}$ with morphisms those 1-cells which have a left adjoint.

3.2.3. THEOREM. *$cod : \mathcal{F}ib_{ladj} \to \mathcal{C}at_{ladj}$ is a subfibration of $cod : \mathcal{F}ib \to \mathcal{C}at$.* In more detail, given $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$, $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{A}\end{smallmatrix}$, *$F \dashv G : \mathbb{A} \to \mathbb{B}$ via η, ϵ and a fibred 1-cell $(\tilde{F}, F) : p \to q$ as shown below*

$$\begin{tikzcd}
\mathbb{E} \arrow[r, "\tilde{F}"] \arrow[d, "p"'] & \mathbb{D} \arrow[d, "q"] \\
\mathbb{B} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{A} \arrow[l, shift left, "G"]
\end{tikzcd}$$

let $\hat{F} : p \to F^*(q)$ in $\mathcal{F}ib(\mathbb{B})$ be the unique mediating functor in

$$\begin{tikzcd}
\mathbb{E} \arrow[dr, "\hat{F}"] \arrow[drr, bend left, "\tilde{F}"] \arrow[ddr, bend right, "p"'] & & \\
& F^*(\mathbb{D}) \arrow[r, "q^*(F)"] \arrow[d, "F^*(q)"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{D} \arrow[d, "q"] \\
& \mathbb{B} \arrow[r, "F"'] & \mathbb{A}
\end{tikzcd}$$

Then, the following are equivalent

$$\begin{array}{rl} (i) & \exists \tilde{G} : \mathbb{D} \to \mathbb{E}. \tilde{F} \dashv \tilde{G}(in\ \mathcal{C}at)\ s.t.\ (\tilde{F}, F) \dashv (\tilde{G}, G) : q \to p(in \mathcal{F}ib). \\ (ii) & \exists \hat{G} : F^*(q) \to p. F \dashv \hat{G}(in\ \mathcal{F}ib(\mathbb{B})). \end{array}$$

Proof. (ii) $\implies$ (i) This implication means that it is possible to define a 'global' fibred right adjoint $\tilde{G}$ given a vertical one $\hat{G}$ and a base one G. This is achieved by composition of adjoints.

By Lemma 3.2.1, we get a fibred right adjoint to F', $F' \dashv \bar{G} : q \to F^*(q)$ via $\overline{\eta}$, $\overline{\epsilon}$, and therefore $\tilde{G} = \hat{G} \circ \bar{G}$ is a right adjoint to $\tilde{F}$. It only remains to verify that the unit $\tilde{\eta} = \hat{G}\overline{\eta}\hat{F} \circ \hat{\eta}$ of this adjunction, where $\hat{\eta}$ is the unit for $\hat{F} \dashv \hat{G}$, is over η:

$$p\tilde{\eta} = p\hat{G}\overline{\eta}\hat{F} \circ p\hat{\eta} = F^*(q)\overline{\eta}\hat{F} = qF^*(q)\hat{F} = \eta p$$

(i) $\implies$ (ii) We first give the intrinsic argument, and then outline the 2-categorical one, as in the proof of Lemma 3.2.1.

Given $(I, X) \in |F^*(\mathbb{D})|$, *i.e.* $qX = FI$, let

$$\hat{G}(I, X) = \eta_I^{*p}(\tilde{G}X)$$

It is the domain of an arbitrary cartesian lifting of $\eta_I : I \to GFI$ at $\tilde{G}X$. The instance of the counit $\hat{\epsilon}_{(I,X)} : \hat{F}\hat{G}(I, X) \to (I, X)$ is given by

$$\hat{\epsilon}_{(I,X)} = \tilde{\epsilon}_X \circ \tilde{F}(\overline{(\eta_I)}^p(\tilde{G}X))$$

as shown in the following diagram

$$\begin{tikzcd}
\tilde{F}(\eta_I^{*p}(\tilde{G}X)) \arrow[r, "\tilde{F}(\overline{(\eta_I)}^p)"] & \tilde{F}\tilde{G}X \arrow[r, "\tilde{\epsilon}_X"] & X \arrow[d, dotted, "q"] \\
FI \arrow[r, "F\eta_I"'] & FGFI \arrow[r, "\epsilon_{FI}"'] & FI
\end{tikzcd}$$

To verify its couniversality, for $Y \in |\mathbb{E}_I|$ a vertical morphism $f : \hat{F}Y \to (I, X)$ induces a unique morphism $f' : Y \to \tilde{G}X$ by the adjunction $\tilde{F} \dashv \tilde{G}$, such that $\tilde{\epsilon} \circ \tilde{F}(f') = f$ and $pf' = \eta_I$, because the adjoint transpose of f across $\tilde{F} \dashv \tilde{G}$ is over the adjoint transpose of 1_{FI} across $\tilde{F} \dashv \tilde{G}$ by the definition of fibred adjunction. So, f' factors through $\overline{(\eta_I)}^p(\tilde{G}X)$, giving a unique vertical morphism $\hat{f} : Y \to \eta_I^{*p}(\tilde{G}X)$ with

$$\tilde{\epsilon}_X \circ \hat{F}(\overline{(\eta_I)}^p(\tilde{G}X)) \circ \hat{F}(\hat{f}) = \tilde{\epsilon}_X \circ \tilde{F}(f') = f$$

as required.

The 2-categorical argument goes as follows: $\eta : 1_{\mathbb{B}} \Rightarrow GF$ induces $\langle \eta \rangle_p : (GF)^*(p) \to p$ by Lemma 3.1.2.(i). Consider the following diagram

$$\begin{tikzcd}
F^*(p) \arrow[r, "q^*(F)"] \arrow[d, "F^*(G')"'] \arrow[dr, phantom, "\lrcorner", very near start] & q \arrow[d, "G'"'] \arrow[dr, "\tilde{G}"] & \\
(GF)^*(p) \arrow[r, "(G^*(p))^*(F)"'] & G^*(p) \arrow[r, "p^*(G)"'] & p
\end{tikzcd}$$

where $G' = \langle q, \tilde{G} \rangle$. Then $\hat{G} = \langle \eta \rangle_p \circ F^*(G')$ is the desired right adjoint. The unit $\hat{\eta} : 1_p \Rightarrow \hat{G}\hat{F}$ is obtained applying Lemma 3.1.2.(ii) to the fibred 2-cell

$(\tilde{\eta}, \eta)$. The counit $\hat{\epsilon} : \hat{F}\hat{G} \Rightarrow 1_{F^*(p)}$ is obtained applying Lemma 3.1.2.(ii) to the fibred 2-cell

$$(\tilde{\epsilon} q^*(F) \circ \tilde{F}\eta'_p F^*(G'), \epsilon F \circ F\eta) : (\tilde{F}\hat{G}, F) \Rightarrow (q^*(F), F)$$

The triangular laws are verified using the algebraic laws for 2-cells of ¶3.1.1, as in the proof of Lemma 3.2.1. □

3.2.4. Remarks.

- (ii) $\Longrightarrow$ (i) above does not require that p be a fibration. Similarly, (i) $\Longrightarrow$ (ii) does not require that q be a fibration.
- Cartesian fibred adjunctions are precisely the Cartesian morphisms for the fibration $cod : \mathcal{Fib}_{ladj} rightarrow \mathcal{Cat}_{ladj}$, which justifies the terminology. Lemma 3.2.1 asserts that pullbacks in $\mathcal{Cat}$ provide a cleavage for this fibration.

By mere duality, we get the following results concerning lifting and factorisation of cofibred adjunctions for cofibrations. Recall that $\_^{op} : \mathcal{Cat} \to \mathcal{Cat}^{co}$ turns right adjoints into left adjoints and vice versa.

3.2.5. Corollary. *Given a cofibration $q : \mathbb{D} \to \mathbb{B}$ and an adjunction $F \dashv G : \mathbb{A} \to \mathbb{B}$ via η, ϵ, change-of-base along G yields a cartesian cofibred adjunction*

Proof. Apply Lemma 3.2.1 to the fibration q^{op} and the adjunction $G^{op} \dashv F^{op}$.

$$\begin{tikzcd}
F^*(\mathbf{E}) \arrow[r, shift left, "q^*(G)"] \arrow[r, phantom, "\top"] \arrow[d, "F^*(q)"'] & E \arrow[l, shift left, "\tilde{F}"] \arrow[d, "q"] \\
\mathbf{A} \arrow[r, shift left, "G"] \arrow[r, phantom, "\top"] & \mathbf{B} \arrow[l, shift left, "F"]
\end{tikzcd}$$

□

3.2.6. Corollary. *cod* : $CoFib_{radj} \to \mathcal{C}at_{radj}$ *is a subfibration of cod* : $CoFib \to \mathcal{C}at$.

Proof. From Theorem 3.2.3, by a duality argument as in Corollary 3.2.5. See Remark 1.1.4. □

In the following section we show two important consequences of Theorem 3.2.3 dealing with co)limits and cartesian closure for fibred categories.

3.3 Fibred limits and cartesian closure

We will apply Theorem 3.2.3 to give a characterisation of the completeness of the total category of a fibration in terms of that of the fibres and of the base category. In order to do so, we shall make use of the following simple property of the exponential 2-functor $(\_)^{\mathbb{I}}$ (for $\mathbb{I}$ a small category) in $\mathcal{C}at$, *i.e.* the 2-functor such that $\mathbb{A}^{\mathbb{I}}$ is the functor category.

3.3.1. Proposition. *Given a fibration* $p : \mathbb{E} \to \mathbb{B}$ *and a small category* $\mathbb{I}, p^{\mathbb{I}} : \mathbb{E}^{\mathbb{I}} \to \mathbb{B}^{\mathbb{I}}$ *is a fibration.*

Proof. A natural transformation $\alpha : F \dot{\to} G : \mathbb{I} \to \mathbb{E}$ is $p^{\mathbb{I}}$-cartesian iff every component is p-cartesian. Thus a $p^{\mathbb{I}}$-cartesian lifting is obtained from p-cartesian liftings, pointwise. $\square$

3.3.2. Remark. The above proposition actually shows that $\mathcal{F}ib$ has *cotensors*, as in $\mathcal{C}at$, in the sense of [Kel89]. This means that we have the following isomorphism of categories

$$\mathcal{F}ib(q, p^{\mathbb{I}}) \cong \mathcal{C}at(\mathbb{I}, \mathcal{F}(q, p))$$

2-natural in q and p.

We shall also use the following property of right adjoints in $\mathcal{C}at/\mathbb{A}$ and $\mathcal{C}^{\to}$. It turns out that such right adjoints preserve cartesian morphisms.

3.3.3. Lemma.

(i) ([Win90, Lemma 4.5]) Given $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$, $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{B}\end{smallmatrix}$ *, a 1-cell* $G : q \to p$ *in* $\mathcal{C}at/\mathbb{B}$*, if there is* $F : p \to q$ *such that* $F \dashv G$ *in* $\mathcal{C}at/\mathbb{B}$ *then* G *is a* $\mathbb{B}$*-fibred 1-cell.*

(ii) Given $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$, $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{A}\end{smallmatrix}$ *, a 1-cell* $(\tilde{G}, G) : q \to p$ *in* $\mathcal{C}al^{\to}$*, if there is* $(\tilde{F}, F) : p \to q$ *such that* $(\tilde{F}, F) \dashv (\tilde{G}, G)$ *in* $\mathcal{C}at$ *then* $(\tilde{G}, G)$ *is a fibred 2-cell.*

Proof.

(i) We simply have to show that G preserves cartesian morphisms. Using Proposition 1.2.17, this amounts to

$$G^{\rightarrow} \circ \overline{(\_)}^{\wedge q} \cong \overline{(\_)}^{\wedge p} \circ cod^*(G)$$

which holds because they have a common left adjoint:

$$G^{\rightarrow} \circ \overline{(\_)}^{\wedge q} \dashv I_p \circ F^{\rightarrow} = cod^*(F) \circ I_q \dashv \overline{(\_)}^{\wedge p} \circ cod^*(G)$$

(ii) Similar argument in $\mathcal{C}at^{\rightarrow}$ instead of $\mathcal{C}at/\mathbb{B}$.

$\square$

The following definition of fibred $\mathbb{I}$-limits is due to Bénabou.

3.3.4. Definition. For any small category $\mathbb{I}$, a fibration $p : \mathbb{E} \rightarrow \mathbb{B}$ has fibred $\mathbb{I}$-limits (respectively colimits) iff the fibred functor $\hat{\Delta}_{\mathbb{I}} : p \rightarrow \Delta^*_{\mathbb{I}}(p^{\mathbb{I}})$, uniquely determined in the diagram below, has a fibred right (respectively left) adjoint $\widehat{\Delta_{\mathbb{I}}} \dashv \widehat{\mathrm{Lim}_{\mathbb{I}}}$

$$\begin{tikzcd}
\mathbb{E} \arrow[dr, "\hat{\Delta}_{\mathbb{I}}"] \arrow[drr, "\tilde{\Delta}_{\mathbb{I}}"] \arrow[ddr, "p"'] & & \\
& \Delta^*_{\mathbb{I}}(\mathbb{E}) \arrow[r, "(p^{\mathbb{I}})^*(\Delta_{\mathbb{I}})"'] \arrow[d, "\Delta^*_{\mathbb{I}}(p^{\mathbb{I}})"] & \mathbb{E}^{\mathbb{I}} \arrow[d, "p^{\mathbb{I}}"] \\
& \mathbb{B} \arrow[r, "\Delta_{\mathbb{I}}"'] & \mathbb{B}^{\mathbb{I}}
\end{tikzcd}$$

where $\Delta_{\mathbb{I}} : \mathbb{B} \rightarrow \mathbb{B}^{\mathbb{I}}$ and $\tilde{\Delta}_{\mathbb{I}} : \mathbb{E} \rightarrow \mathbb{E}^{\mathbb{I}}$ are the diagonal functors taking objects A to constant functors $(I \mapsto A)$.

Dually, we speak of fibred $\mathbb{I}$-colimits, and of cofibred $\mathbb{I}$-limits/colimits for a cofibration.

3.3.5. Remarks.

- Similarly to Remark 3.3.2, the fibration $\Delta_{\mathbb{I}}^*(p^{\mathbb{I}})$ is a cotensor in $\mathcal{F}ib(\mathbb{B})$, as we have

$$\mathcal{F}ib(\mathbb{B})(q, \Delta_{\mathbb{I}}^*(p^{\mathbb{I}})) \cong \mathcal{C}at(\mathbb{I}, \mathcal{F}ib(\mathbb{B})(q,p))$$

  Hence, the above definition of fibred $\mathbb{I}$-limits for a fibration is analogous to the definition of $\mathbb{I}$-limits for an ordinary category. Remember that a category $\mathbb{C}$ has $\mathbb{I}$-limits if the diagonal $\Delta_{\mathbb{I}} : \mathbb{C} \to \mathbb{C}^{\mathbb{I}}$ has a right adjoint.

- Using the indexed view of fibred adjunctions, a fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ has fibred $\mathbb{I}$ limits if every fibre has $\mathbb{I}$-limits, in their usual sense in $\mathcal{C}at$, and the reindexing functors are $\mathbb{I}$-continuous, *i.e.* preserve $\mathbb{I}$-limits.

Now we can characterise fibred limits as follows:

3.3.6. Corollary. *Let $\mathbb{I}$ be a small category and $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ be a fibration such that $\mathbb{B}$ has $\mathbb{I}$-limits. Then p has fibred $\mathbb{I}$-limits iff $\mathbb{E}$ has and p strictly preserves $\mathbb{I}$-limits.*

Proof. Apply Theorem 3.2.3 to the following data (where $p^{\mathbb{I}} : \mathbb{E}^{\mathbb{I}} \to \mathbb{B}^{\mathbb{I}}$ is a fibration by Proposition 3.3.1)

$\mathbb{E}$ has and p strictly preserves $\mathbb{I}$-limits means precisely that the above diagram

$$\begin{tikzcd}
\mathbb{E} \arrow[r, "\tilde{\Delta}_{\mathbb{I}}"] \arrow[d, "p"'] & \mathbb{E}^{\mathbb{I}} \arrow[d, "p^{\mathbb{I}}"] \\
\mathbb{B} \arrow[r, shift left, "\Delta_{\mathbb{I}}"] \arrow[r, phantom, "\perp"] & \mathbb{B}^{\mathbb{I}} \arrow[l, shift left, "\mathrm{Lim}_{\mathbb{I}}"]
\end{tikzcd}$$

can be completed to an adjunction $(\tilde{\Delta}_{\mathbb{I}}, \Delta_{\mathbb{I}}) \dashv (\tilde{\mathrm{Lim}}_{\mathbb{I}}, \mathrm{Lim}_{\mathbb{I}})$ in $\mathcal{C}at^{\rightarrow}$, which by Lemma 3.3.3.(ii) is an adjunction in $\mathcal{F}ib$. □

3.3.7. COLLARY. *Let $r : \mathbb{D} \to \mathbb{A}$ be a cofibration such that $\mathbb{A}$ has $\mathbb{I}$-colimits. Then r has cofibred $\mathbb{I}$-colimits iff $\mathbb{D}$ has and r strictly preserves $\mathbb{I}$-colimits.*

3.3.8. REMARKS.

- Corollary 3.3.6 yields a stronger version of [BGT91, Theorem 1].

- Recall that a fibration is a bifibration (*i.e.* its dual is a fibration as well) iff every reindexing functor has a left adjoint, *cf.* Proposition 1.2.7.(iii). Thus Theorem 2 in *ibid.* is a consequence of Corollary 3.3.7.

- *ibid.* applies these results about fibred limits and colimits to prove (co)completeness of several categories of relevance in the area of algebraic specifications, such as those in Examples 1.3.3.

We make explicit the expressions for finite products in $\mathbb{E}$ obtained by lifting ('only if' direction of Corollary 3.3.6) in the following definition. We shall make use of this description in Corollary 3.3.11 below and in §4, which will justify the terminology to be introduced.

3.3.9. DEFINITION (logical finite products). Given $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ fibred-cc, where $\mathbb{B}$ has finite products. Then, by Corollary 3.3.6 $\mathbb{E}$ has

(i) *logical terminal object* $\tilde{1} \cong 1(1)$ (the terminal object of $\mathbb{E}_1$). For X in $\mathbb{E}_A$, $\tilde{!}_X \cong \overline{!_A}(1(1))\circ !(1)_X : X \to 1(1)$, where $!_A$ is the unique morphism $A \to 1$ in $\mathbb{B}$ and $!(1)_X$ is the unique morphism $X \to 1(A) \cong (!_A)^*(1(1))$.

(ii) *logical binary products:* for $X \in |\mathbb{E}_A|, Y \in |\mathbb{E}_B|$,

$$X \tilde{\times} Y \cong (\pi_{A,\mathbb{B}})^*(X) \times_{A\times\mathbb{B}} (\pi'_{A,\mathbb{B}})^*(Y)$$

where $\times_{A\times\mathbb{B}}$ is the product in the fibre $\mathbb{E}_{A\times\mathbb{B}}$. Projections:

$$\tilde{\pi}_{X,Y} = \overline{\pi_{A,\mathbb{B}}}(X) \circ \pi_{\pi^*(X),(\pi')^*(Y)} : X \times Y \to X$$

(where the second projection is taken in the fibre $\mathbb{E}_{A\times\mathbb{B}}$), and a similar expression for $\tilde{\pi}'_{X,Y}$.

3.3.10. DEFINITION (logical finite coproducts). Dually, assume that $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ has fibred finite coproducts and cocartesian liftings along coproduct injections. By Corollary 3.3.7, $\mathbb{E}$ has

(i) *logical initial object* $\tilde{0} \cong 0(0)$ (the initial object of $\mathbb{E}_0$).

(ii) *logical binary coproducts*: for $X \in |\mathbb{E}_A|$ and $Y \in |\mathbb{E}_\mathbb{B}|$,

$$X \tilde{+} Y \cong (\iota'_{A,\mathbb{B}})_! Y$$

with injections

$$\iota_{X,Y} = \iota_{(\iota_{A,\mathbb{B}})_! X, (\iota'_{A,\mathbb{B}})_! Y} \circ \underline{\iota}_{A,\mathbb{B}}((\iota_{A,\mathbb{B}})_! X)$$

where the first injection is in the fibre $\mathbb{E}_{A+\mathbb{B}}$, and a similar expression for $\tilde{\iota}'_{X,Y}$.

Logical products and coproducts will be used in §4.5 to formulate the induction principle for inductive data types in a distributive category.

Another useful consequence of Theorem 3.2.3 is the following sufficient condition to lift cartesian closed structure. This result can be seen as a categorical version of logical predicates. This will be explained in detail in §4.2.

3.3.11. Corollary. *Given* $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{array}$ *such that p is a fibred-ccc with $Cons_{\mathbb{B}}$-products, if $\mathbb{B}$ is a ccc then $\mathbb{E}$ is a ccc and p strictly preserves the cartesian closed structure.*

Proof. From Corollary 3.3.6 we know $\mathbb{E}$ has finite products. Then, for every $X \in \mathbb{E}_A$, we must supply $X \tilde{\Rightarrow} \_ : \mathbb{E} \to \mathbb{E}$ such that the following is a fibred adjunction:

$$\begin{tikzcd}
\mathbf{E} \arrow[r, shift left=2, "_\tilde{\times} X"] \arrow[r, phantom, "\perp"] \arrow[d, "p"'] & \mathbf{E} \arrow[l, shift left=2, "X \tilde{\Rightarrow} _"] \arrow[d, "p"] \\
\mathbf{B} \arrow[r, shift left=2, "_ \times A"] \arrow[r, phantom, "\perp"] & \mathbf{B} \arrow[l, shift left=2, "A \Rightarrow _"]
\end{tikzcd}$$

By Theorem 3.2.3, it is sufficient to define a fibred right adjoint G to $\overline{\_\tilde{\times} X}$, as displayed below

$$\begin{tikzcd}
\mathbf{E} \arrow[r, shift left=2, "\overline{_\tilde{\times} X}"] \arrow[r, phantom, "\perp"] \arrow[dr, "p"'] & \mathbf{E} \bullet A \arrow[l, shift left=2, "G"] \arrow[r] \arrow[d, "p_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E} \arrow[d, "p"] \\
 & \mathbf{B} \arrow[r, "_ \times A"'] & \mathbf{B}
\end{tikzcd}$$

If we examine the action of $\overline{\_\tilde{\times} X}$ on a particular fibre $\mathbb{E}_C$ (*cf.* Definition 3.3.9), we see that it can be factored in the following way:

$$\begin{tikzcd}
\mathbf{E}_C \arrow[rr, "\overline{_\tilde{\times} X}|_C"] \arrow[dr, "\pi^*_{C,A}"'] & & \mathbf{E}_{C \times A} \\
 & \mathbf{E}_{C \times A} \arrow[ur, "_ \times^{C \times A} (\pi')^* X"'] &
\end{tikzcd}$$

Then, we have $(\_ \times^{C\times A} (\pi')^* X) \dashv ((\pi')^* X \Rightarrow^{C\times A} \_)$ and $\pi^*_{C,A}$ so, we have a family of right adjoints $G_C = \Pi_A \circ ((\pi')^* X \Rightarrow^{C\times A})$; since p is a fibred ccc and Π_A is a $Cons_A$-product, such a family underlies a fibred right adjoint $\overline{\_ \tilde{\times} X} \dashv_f G$ as desired (using [Jac91a, Lemma 1.2.2] and Proposition 1.4.5). $\square$

We spell out the expression for exponentials and evaluation morphisms in the following definition. The terminology will be justified in §4.2.

3.3.12. Definition. Given $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ satisfying the hypothesis of Corollary 3.3.11, $\mathbb{E}$ has logical exponents: for $X \in |\mathbb{E}|_A$, $Y \in |\mathbb{E}|_{\mathbb{B}}$,

$$X \tilde{\Rightarrow} Y \cong \Pi_A((\pi'_{A\Rightarrow\mathbb{B},A})^*(X) \Rightarrow_{A\Rightarrow\mathbb{B}\times A} \mathrm{ev}^*_{A,\mathbb{B}}(Y))$$

where $\Rightarrow_{A\Rightarrow\mathbb{B}\times A}$ is the exponential functor in the fibre $\mathbb{E}_{A\Rightarrow\mathbb{B}\times A}$ and $\mathrm{ev}_{A,\mathbb{B}}$: $A \Rightarrow B \times A \to B$ is the B-component of the counit of $(\_) \times A \dashv A \Rightarrow (\_)$. The *logical evaluation morphism* is

$$\tilde{\mathrm{ev}}_{X,Y} \cong \overline{\mathrm{ev}_{A\Rightarrow B,A}}(Y) \circ \mathrm{ev}_{((\pi')^*(X)\Rightarrow \mathrm{ev}^*(Y),(\pi')^*(X))} \circ \epsilon_{(\pi')^*(X)\Rightarrow \mathrm{ev}^{ast}(Y)} : \\ X \tilde{\Rightarrow} Y \tilde{\times} X \to Y$$

where ϵ is the counit of $(\pi'_{A\Rightarrow B,A}) \dashv \Pi_A$.

We end this chapter with a concrete simple example of Corollary 3.3.11. The concrete description of the 'logical' cartesian closed structure in this example has a suggestive shape, which hints at the connection with logical predicates which we will make explicit in §4.2.

3.3.13. Example. Consider $\iota : Sub(\mathcal{S}et) \to \mathcal{S}et$. $\mathcal{S}et$ is cartesian closed and ι is a fibred-ccc as mentioned in Ex. 1.4.3.(iii): products in a fibre $Sub(\mathcal{S}et)_X \cong \mathbf{195}X$ are given by intersections, while exponentials are given by

$$(S \subseteq X) \Rightarrow (S' \subseteq X) \cong \{x \in X \mid x \in s \Longrightarrow x \in S'\}$$

It also has $Cons_{\mathcal{S}et}$-products, cf Example 1.4.6.(iii).

Products and exponentials in $Sub(\mathcal{S}et)$ can be described as follows. Given $S_A \subseteq A$ and $S_B \subseteq B$, we have

$$\begin{aligned} S_A \tilde{\times} S_B &\cong \{(x,y) \in A \times B \mid x \in S_A \wedge y \in S_B\} \\ S_A \tilde{\Rightarrow} S_B &\cong \{f : A \to B \mid \forall x \in A. x \in S_A \Rightarrow fx \in S_B\} \end{aligned}$$

because

$$\begin{aligned} (\pi'_{A \Rightarrow B, A})^*(S_A) &\cong \{(f,x) \in (A \Rightarrow B) \times A \mid x \in S_A\} \\ \mathrm{ev}^*_{A,B}(S_B &\cong \{(f,x) \in (A \Rightarrow B) \times A \mid fx \in S_B\} \end{aligned}$$

Chapter 4

Logical predicates for simply typed λ-calculus

The material about fibrations of §3, notably Corollary 3.3.11, is a basis for a category-theoretic account of logical predicates for simply typed λ-calculus, based on the correspondence between λ-calculus and cartesian closed categories as in §2.1.1.

In §4.1 we introduce the internal language of a fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ satisfying the hypothesis of Corollary 3.3.11. By expressing the 'logical' cartesian closed structure of $\mathbb{E}$, as detailed in Definitions 3.3.9 and 3.3.12, in this language, we obtain the formulas corresponding to logical predicates for simply typed λ-calculus. We also show how the essential property of logical predicates, namely the Basic Lemma 2.2.4, results from expressing in the internal language the soundness of typing for the interpretation of λ-calculus in $\mathbb{E}$.

In §4.3 we present several examples of fibred-ccc's with products. First, we consider the injective scone of a category as given in [MS92], which captures logical predicates for applicative structures as in §2.2. In a similar way, we get admissible logical predicates for $\omega\mathcal{C}po$. A further example of logical predicates is that of Kripke logical predicates, as in [MM91]. A different kind of example is provided by the category of first-order deliverables, introduced in [BM91] to structure program development in type theory; its cartesian closed structure follows from Corollary 3.3.11. As a final example, we show how to infer the cartesian closed structure of ω-$\mathcal{S}et$ and PER from the above Corollary and the properties of reflective and coreflective categories in §2.3.

In §4.4, we comment on the relationship between our approach to logical predicates and that in [MR91].

In §4.5, we give a categorical formulation of the induction principle for inductive data types in a distributive category. The approach follows that of logical predicates, namely exploiting the logical meaning of the structure of the total category of a fibration via its internal language.

4.1 Internal language for a fibred-ccc with products

Let $\begin{smallmatrix}\mathbb{H}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ be a fibred-ccc with $Cons_{\mathbb{B}}$-products and $\mathbb{B}$ a ccc. For instance, every first-order hyperdoctrine, *cf.* Definition 2.1.1, is such. We will define its *iternal language* in the usual categorical-logic style, as in [LS86, Part I, §10.6]. For instance, the internal language of a ccc $\mathbb{C}$ is the simply typed λ-calculus whose types are objects of $\mathbb{C}$, terms are morphisms of $\mathbb{C}$ and equations between them reflect the equality of morphisms in $\mathbb{C}$.

The internal language of $\begin{smallmatrix}\mathbb{H}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ is the $\{\forall, \Longrightarrow, \wedge, \top\}$-fragment of first-order intuitionistic predicate calculus as outlined in §2.1.2, whose types, propositions, terms and equations are determined by $\begin{smallmatrix}\mathbb{H}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$. In more detail, the theory $\mathcal{T}$ has a type, $\vdash A\ \mathsf{Type}$, for every object A of $\mathbb{B}$ and a term

$$\frac{}{x : A \vdash u : B}$$

for every morphism $u : A \to B$ in $\mathbb{B}$, with the appropriate equations between terms corresponding to the equality of morphisms in $\mathbb{B}$. Since $\mathbb{B}$ is a ccc, such equations include those of the simply typed λ-calculus. We have the following correspondence between substitution of terms for variables in the language and composition of morphisms in $\mathbb{B}$:

$$\frac{x : B \vdash u : A \quad \Theta \vdash v : B}{\Theta \vdash u[x := v] = u \circ v : A}$$

Propositions in context, or predicates, correspond to objects of $\mathbb{H}$; for every $P \in |\mathbb{H}_A|$ there is a judgement

$$\frac{}{x : A \vdash P(x)\,\mathsf{Prop}}$$

That is, objects of $\mathbb{H}_A$ correspond to propositions in context $x : A$ in the internal language. So, in every context Θ we have the simply typed λ-calculus of propositions and proofs in such context, corresponding to the internal language of $\mathbb{H}_\Theta$, which is a ccc. Thus, there are rules relating the logical connectives with operations on the fibres:

$$\frac{}{x : A \vdash 1_A \leftrightarrow \top} \qquad \frac{x : A \vdash P\,\mathsf{Prop} \qquad x : A \vdash Q\,\mathsf{Prop}}{x : A \vdash P \times_A Q \leftrightarrow P \wedge Q}$$

$$\frac{x : A \vdash P\,\mathsf{Prop} \qquad x : A \vdash Q\,\mathsf{Prop}}{x : A \vdash P \Rightarrow_A Q \leftrightarrow P \Longrightarrow Q}$$

where, in general, $\Theta \vdash P \leftrightarrow Q$ indicates that there is a canonical isomorphism between P and Q in context Θ, which we leave implicit to avoid notational clutter. The subscript $_A$ indicates that the operations are those of the fibre $\mathbb{H}_A$.

Additionally, since we are dealing with a fibration, there are rules for changing contexts. There are two operations we can perform: reindexing, which corresponds to substitution of terms for variables in predicates and proofs, and the $Cons_{\mathbb{B}}$-products, which correspond to universal quantification. For $u : A \to B$, $u^* : \mathbb{H}_B \to \mathbb{H}_A$ corresponds to substitution in the internal language:

$$\frac{x : A \vdash u : B \qquad y : B \vdash P(y)\,\mathsf{Prop}}{x : A \vdash u^*(P) \leftrightarrow P(u)}$$

Likewise, the correspondence between the $Cons_{\mathbb{B}}$-products and universal quantification is expressed by

$$\frac{\Theta, x : A \vdash P\,\mathsf{Prop}}{\Theta \vdash \Pi_A P \leftrightarrow \forall x : A.P}$$

A morphism $f : P \to Q$ in $\mathbb{H}$, with $pf = u : A \to B$, can be identified with its vertical factor $\hat{f} : P \to u^*(Q)$ in $\mathbb{H}_A$. Hence, in the internal language f corresponds to

$$\frac{}{x : A \mid h : P \vdash \hat{f} : Q(u)}$$

$\hat{f}$ corresponds to a proof that the hypothesis P entails $Q(u)$. The above correspondence amounts to considering the equivalent fibration resulting from the Grothendieck construction applied to the indexed category induced by the chosen cleavage for p, whereby we identify f with $(\hat{f}, u)$, *cf.* Proposition 1.3.6.

4.2 Logical predicates for cartesian closed categories

In §2.2, we presented logical predicates for set-theoretic models of simply typed λ-calculus. We now present them for categorical models, *i.e.* cartesian closed categories. We first show precisely how logical predicates for simply typed λ-calculus arise by interpreting the logical finite products and exponentials of Definitions 3.3.9 and 3.3.12 respectively, in the internal language of a fibration with suitable structure, as presented in §4.1.

Let $\mathbb{B}$ be a ccc, regarded as a model of a simply typed λ-calculus. A fibration $\begin{smallmatrix}\mathbb{H}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$, which is a fibred-ccc and has $Cons_{\mathbb{B}}$-products, can be regarded, via its internal language, as a first order logic over $\mathbb{B}$, with $\{\forall, \Longrightarrow, \wedge, \top\}$ as logical symbols, as outlined in the previous section. The cartesian closed structure of $\mathbb{H}$ is expressed in this language as follows:

Terminal object in $\mathbb{H}$:

$$\tilde{1} \triangleq x : 1 \vdash \top \mathsf{Prop}$$

Binary product in $\mathbb{H}$: For $P \in \mathbb{H}_A$ and $Q \in \mathbb{H}_B$,

$$P \tilde{\times} Q \triangleq z : A \times B \vdash P(\pi z) \wedge Q(\pi' z) \mathsf{Prop}$$

and the projections $P \overset{\tilde{\pi}}{\leftarrow} P \tilde{\times} Q \overset{\tilde{\pi}'}{\rightarrow} Q$, which are over $A \overset{\pi}{\leftarrow} A \times B \overset{\pi'}{\rightarrow} B$ correspond to proofs

$$z : A \times B \mid p : P(\pi z) \wedge Q(\pi' z) \vdash \hat{\pi} : P(\pi z)$$

and

$$z : A \times B \mid p : P(\pi z) \wedge Q(\pi' z) \vdash \hat{\pi}' : Q(\pi' z)$$

respectively.

Exponentials in $\mathbb{H}$: For $P \in \mathbb{H}_A$ and $Q \in \mathbb{H}_B$,

$$P \Rightarrow Q \triangleq f : A \Rightarrow B \vdash \forall x : A.(P(x) \Longrightarrow Q(\mathrm{ev}_{A,B}\langle f, x\rangle))$$

because, for a predicate $f : A \Rightarrow B, x : A \vdash Q\mathsf{Prop}$

$$f : A \Rightarrow B \vdash (\Pi_A Q)(f) \leftrightarrow \forall x : A.Q(f, x)$$

and

$$\begin{array}{l} f : A \Rightarrow B, x : A \vdash \\ ((\pi'_{A\Rightarrow B,A})^*(P) \Rightarrow_{A\Rightarrow B\times A} \quad \mathrm{ev}^*_{A,B}(Q))(f, x) \leftrightarrow P(x) \Longrightarrow Q(\mathrm{ev}_{A,B}(f, x)) \end{array}$$

The evaluation morphism $\tilde{\mathrm{ev}} : P \tilde{\Rightarrow} Q \times P \rightarrow Q$, over $\mathrm{ev}_{A,B} : A \Rightarrow B \times A \rightarrow B$, corresponds to a proof (given by its vertical factor, as explained at the end of the previous section)

$$\begin{array}{l} f : A \Rightarrow B, y : A \mid p : \forall x : \\ A.(P(x) \Longrightarrow Q(\mathrm{ev}\langle f, x\rangle)) \wedge P(y) \vdash \hat{\mathrm{ev}} : Q(\mathrm{ev}\langle f, y\rangle)) \end{array}$$

Just for illustration, let us show how logical predicates for $+$-types can be obtained in this setting. Assume that $\mathbb{B}$ has binary coproducts. In the internal language of $\mathbb{B}$, they correspond to $+$-types, with rules

$$\frac{A\ \mathsf{Type} \quad B\ \mathsf{Type}}{A + B\ \mathsf{Type}}$$

and term forming operators

$$\frac{\Theta \vdash t : A}{\Theta \vdash \iota_{A,B} t : A + B} \qquad \frac{\Theta \vdash t : B}{\Theta \vdash \iota'_{A,B} t : A + B} \qquad \frac{\Theta, x : A \vdash t : C \quad \Theta, y : B \vdash t' : C}{\Theta, Z : A + B \vdash [t, t'] : C}$$

Assume $\begin{smallmatrix}\mathbb{H}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ has fibred coproducts and cocartesian liftings for the coproduct injections $A \xrightarrow{\iota} A + B \xleftarrow{\iota'} B$. By Corollary 3.3.7, $\mathbb{H}$ has coproducts. To spell them out neatly in the internal language of p, we will assume some further conditions on p.

Recall from Proposition 1.2.7.(iii), that the cocartesian liftings for the injections amount to the existence of left adjoints, $\iota_! \dashv \iota^* : \mathbb{H}_{A+B} \to \mathbb{H}_A$ and $\iota'_! \dashv (\iota')^* : \mathbb{H}_{A+B} \to \mathbb{H}_B$. Following [Law70], we express these left adjoints in the internal language of p, assuming an equality predicate at type $A + B$, written $=_{A+B}$ or simply $=$, and $\mathit{Cons}_{\mathbb{B}}$-sums, which correspond to existential quantifiers, *cf.* §2.1.2. We then have

$$\frac{x : A \vdash P}{z : A + B \vdash \iota_!(P) \leftrightarrow \exists x : A.(\iota x = z \wedge P(z))}$$

with an analogous expression for $\iota'_!$.

Recall from Definition 3.3.10 that binary logical coproducts in $\mathbb{H}$ are given as follows: for $P \in \mathbb{H}_A$ and $Q \in \mathbb{H}_B$,

$$P \tilde{+} Q \cong \iota_!(P) +_{A+B} \ \iota'_!(Q)$$

which expressed in the internal language of $\begin{smallmatrix}\mathbb{H}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ become

$$P \tilde{+} Q \triangleq z : A + B \vdash (\exists x : A.(\iota x = z) \wedge P(x)) \vee (\exists y : B.(\iota' y = z) \wedge Q(y)) \,\mathsf{Prop}$$

This is just the expected definition by cases of a logical predicate for a $+$-type.

4.2.1. Remark. The above mentioned equality predicate for a type $\begin{smallmatrix}\mathbb{H}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ at a type $A \in |\mathbb{B}|$ amounts, in categorical terms, to the existence of cocartesian liftings for the diagonal $\delta_A : A \to A \times A$, satisfying appropriate stability conditions. See [Law70] for details.

The above considerations show how certain categorical structure in $\mathbb{H}$, cartesian closure for instance, can be expressed by logical formulas which correspond to logical predicates for the relevant type constructor, *i.e.* products and exponentials. The neat connection between the logical expression, *i.e.* in the internal language, of categorical structure in $\mathbb{H}$ and logical predicates arises because $\mathbb{H}$ is *fibred* and thus we can regard its objects as predicates, and its morphisms and terms and proofs, as we did above.

We recall from [LS86, Part I,§11] that a simply typed λ-calculus $\mathcal{L}$, specified by t its types, terms and equations, generates a cartesian closed category $\mathbb{C}(\mathcal{L})$. It is a term-model construction. Then, an interpretation $[\![\_]\!]$ of $\mathcal{L}$ in a ccc $\mathbb{B}$ corresponds to a functor $[\![\_]\!] : \mathbb{C}(\mathcal{L}) \to \mathbb{B}$ which preserves cartesian closed structure.

Given $\begin{smallmatrix}\mathbb{H}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$, with $\mathbb{H}$ a ccc and p strictly preserving the cartesian closed structure, for instance when p satisfies the hypothesis of Corollary 3.3.11, an interpretation $[\![\tilde{\_}]\!] : \mathbb{C}(\mathcal{L}) \to \mathbb{H}$ of $\mathcal{L}$ in $\mathbb{H}$ yields an interpretation $[\![\_]\!]$ in $\mathbb{B}$, $[\![\_]\!] = p \circ [\![\tilde{\_}]\!] : \mathbb{C}(\mathcal{L}) \to \mathbb{B}$. Regarding $\mathbb{H}$ as a 'category of predicates' over $\mathbb{B}$, the interpretation $[\![\tilde{\_}]\!]$ assigns to a type τ a predicate $[\![\tilde{\tau}]\!]$ over $[\![\tau]\!]$, *i.e.* in the internal language

$$x : \tau \vdash P_x(x)\,\mathsf{Prop}$$

Then, the type-indexed collection $\{P_\tau\}$ is a logical predicate over $\{[\![\tau]\!]\}$. This leads us to the following definition:

4.2.2. Definition. Let $\begin{smallmatrix}\mathbb{H}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ be a fibration with $\mathbb{H}$ and $\mathbb{B}$ ccc, and p strictly preserving the cartesian closed structure. Given an interpretation $\mathcal{A} : \mathbb{C}(\mathcal{L}) \to \mathbb{B}$, a *ccc-logical predicate* $\mathcal{P}$ on $\mathcal{A}$ w.r.t. p is a functor $\mathcal{P} : \mathbb{C}(\mathcal{L}) \to \mathbb{H}$ which preserves cartesian closed structure and $p \circ \mathcal{P} = \mathcal{A}$

4.2.3. Remarks.

- The above definition is the categorical version of Definition 2.2.3; we used $\mathcal{A}$ for interpretations and $\mathcal{P}$ for logical predicates to make the correspondence more evident. Note that the fibration p, which is the logic under consideration is a parameter in the above definition. Indeed, it is possible to have several logics over the same base category, see §4.3.2 for instance.

- Although the set-theoretic definition considers only the object part of $\mathcal{P} : \mathbb{C}(\mathcal{L}) \to \mathbb{H}$ as a logical predicate, the considerations in §4.2.1 below will show that the morphism part of such a functor should be part of the logical predicate as well.

- Notice that the ccc structure is also a parameter in the above definition. As stated, the definition is suitable for simply typed λ-calculus, but we can adapt it to the particular theory under consideration, *e.g.* finite products for algebraic theories, distributivity for inductive data types (*cf.* §4.5).

4.2.1 Basic lemma for categorical logical predicates

The essential property of a (set-theoretic) logical predicate is the so-called Basic Lemma 2.2.4. We will show that for logical predicates for cartesian closed categories, as in Definition 4.2.2, this is an immediate consequence of the *soundness of typing* for the interpretation of simply typed λ-calculus in cartesian closed categories. By soundness of typing we mean that for a term $x : \sigma \vdash t : \tau$, its interpretation in a ccc is a morphism with appropriate domain and codomain, *i.e.* $[\![t]\!] : [\![\sigma]\!] \to [\![\tau]\!]$.

Thus, given a logical predicate $\mathcal{P} : \mathbb{C}(\mathcal{L}) \to \mathbb{H}$ over $\mathcal{A} : \mathbb{C}(\mathcal{L}) \to \mathbb{B}$ a term $x : \sigma \vdash t : \tau$ corresponds under $\mathcal{P}$ to a morphism $\mathcal{P}(t) : \mathcal{P}_\sigma \to Q_\tau$ over $\mathcal{A}(t) : A_\sigma \to A_\tau$. As we mentioned at the end of §4.1, we may identify $\mathcal{P}(t)$ with its vertical factor. We then have as an immediate consequence of Definition 4.2.2.

4.2.4. Corollary *(Basic Lemma for categorical logical predicates). Given a ccc-logical predicate $\mathcal{P} : \mathbb{C}(\mathcal{L}) \to \mathbb{H}$ over $\mathcal{A} : \mathbb{C}(\mathcal{L}) \to \mathbb{B}$, for any term*

$x : \sigma \vdash t : \tau$, *there is a proof* $\hat{t}$

$$x : \sigma \mid h : P_\sigma(x) \vdash \hat{t} : P_\tau(\mathcal{A}(t))$$

where $\hat{t}$ *is given by the vertical factor of* $\mathcal{P}(t)$.

This shows the role of the morphism part of a (categorical) logical predicate: it amounts to a proof of the Basic Lemma for the predicate. We mean proof in a constructive sense, whereby we identify proofs with vertical morphisms, as indicated at the end of §4.1. Of course, in a proof-irrelevant setting, there is at most one such proof and only its existence matters. This is the case for set-theoretic logical predicates, as in §2.2 and §4.3.1. In this case, a logical predicate is determined by the object part of the functor $\mathcal{P} : \mathbb{C}(\mathcal{L}) \to \mathbb{H}$, as considered in [MS92]. Thus, in the case of logical predicates over applicative structures, §2.2, just like a choice of values for constants determines the interpretation of a term in an environment (provided the theory is freely generated), the conditions on Definition 2.2.3 determine the interpretation of a term in the category of predicates. The latter corresponds then to a proof (the unique one in this case) of the basic lemma for models, Lemma 2.2.4.

Notice that Corollary 4.2.4 relies only on the fact that $\begin{smallmatrix}\mathbb{H}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ is a fibration and is independent of the structure of $\mathbb{H}$ and $\mathbb{B}$. It therefore applies to other kind of logical predicates and not only those for simply typed λ-calculus, *cf.* Remarks 4.2.3.

Finally, let us remark that the approach to logical predicates we presented above can deal with n-ary relations as well, by considering fibrations over an n-ary product of categories $p : \mathbb{E} \to \mathbb{B}_1 \times \ldots \times \mathbb{B}_n$; in the internal language of p, the objects of $\mathbb{E}$ correspond to n-ary relations $x_1 : A_1, \ldots , x_n : A_n \vdash R(x_1, \ldots , x_n)$ Prop. In particular, to consider n-ary relations over a given category $\mathbb{B}$ with finite products, given $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$, we consider the fibration p' obtained by change-of-base

$$
\begin{tikzcd}
\mathbf{E}' \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E} \arrow[d, "p"] \\
\mathbf{B}^n \arrow[r, "\times_n"'] & \mathbf{B}
\end{tikzcd}
$$

where $\times_n : \mathbb{B}^n \to \mathbb{B}$ is the n-ary product functor. Thus, objects of $\mathbb{E}'$ correspond to predicates on n variables, or equivalently, n-ary relations on $\mathbb{B}$.

4.3 Some examples

We present examples of fibrations in which Corollary 3.3.11 can be applied. These examples have appeared in the literature, although not explicitly recognised as instances of the abovementioned corollary. They show how diverse categories have cartesian closed structure for the same abstract reason and shows the applicability of our constructions. Further examples of this kind of fibrations are in [Jac92, Jac91a].

4.3.1 Sconing

The fibration $\imath : Sub(\mathcal{S}et) \to \mathcal{S}et$ has the appropriate structure to interpret firstorder predicate calculus, *cf.* Example 2.1.2. Given a category $\mathbb{C}$ with a terminal object 1, we obtain a fibration over it by change-of-base along the global sections functor $\mathbb{C}(1,\_) : \mathbb{C} \to \mathcal{S}et$:

$$\begin{tikzcd} \widetilde{\mathbb{C}} \arrow[r] \arrow[d, "\widetilde{cod}"'] \arrow[dr, phantom, "\lrcorner", very near start] & Sub(\mathcal{S}et) \arrow[d, "\imath"] \\ \mathbb{C} \arrow[r, "\mathbb{C}(1,\_)"'] & \mathcal{S}et \end{tikzcd}$$

$\widetilde{\mathbb{C}}$ is called the *injective scone* of $\mathbb{C}$ in [MS92].

We thus interpret logical formulas in $\widetilde{cod}$ classically, because predicates are interpreted as subsets. $\widetilde{cod}$ is a fibred-ccc, since $\mathbb{C}(1,\_)^* : \mathcal{F}ib\mathcal{S}et \to \mathcal{F}ib\mathbb{C}$ is a 2-functor which preserves finite products, by Lemma 3.1.1, and hence preserves the relevant adjunctions. It also has $Cons_{\mathbb{C}}$ products by Proposition 1.4.7. Thus, when $\mathbb{C}$ is a ccc, so is $\widetilde{\mathbb{C}}$ and $\widetilde{cod}$ preserves such structure, by Corollary 3.3.11.

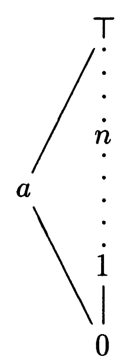
The expression of the logical cartesian closed structure in such a fibration then corresponds to classical logical predicates on $\mathbb{C}$, as in [MS92], or rather, to logical predicates on the applicative structure generated by $\mathbb{C}$, *cf.* Remark 2.2.2. Specifically, given objects $A, B \in |\mathbb{C}|$ and subsets $R \subseteq \mathbb{C}(1, A)$ and $S \subseteq \mathbb{C}(1, B)$, their exponential in $\widetilde{\mathbb{C}}$ is: for $f : 1 \to A \Rightarrow B$

$$f \in (R \mathrel{\dot{\Rightarrow}} S) \Leftrightarrow \forall x : 1 \to A. x \in R \Longrightarrow \mathrm{ev}_{A,B}\langle f, x\rangle \in S$$

In certain cases we may transfer structure from the injective scone of a category to its category of subobjects (or rather, from the classical logic over it to its internal logic). This is the case with $\omega\mathcal{C}po$ which we analyse below.

4.3.2 Logical predicates for complete partial orders

Let $\omega\mathcal{C}po$ be the category of ω-complete posets (not necessarily with a bottom element) and continuous functions between them. It is a standard tool in denotational semantics, see *e.g.* [LS81]. Consider the fibration $\imath : Sub'(\omega\mathcal{C}po) \to \omega\mathcal{C}po$, where an object in $Sub'(\omega\mathcal{C}po)_C$ is a subset of C closed under sups of ω-chains, a so-called *admissible subset*, and morphisms are commutative squares. Thus, every fibre $(Sub'(\omega\mathcal{C}po))_C$ is small complete, with limits given by intersection. But the fibres are not cartesian closed. For a counterexample, consider the cpo



Consider the admissible subset $\{0, a, \top\}$ and the family of admissible subsets $\{0, \ldots, i\}$, for $i \in \omega$. Then

$$\{0, a, \top\} \cap (\bigvee_{i\in\omega} \{0, \ldots, i\}) = \{0, \top\} \neq \{0\} = \bigvee_{i\in\omega} (\{0, a, \top\} \cap \{0, \ldots, i\})$$

This shows that $\{0, a, \top\} \cap \_$ does not preserve colimits and therefore cannot have a right adjoint.

So, we cannot apply Corollary 3.3.11 directly to this fibration in order to get logical predicates for this logic over $\omega\mathcal{C}po$. However, we can use sconing to get around them.

Consider the following change-of-base diagram

$$\begin{tikzcd} \kappa\mathcal{C}po \arrow[r, "\imath^*(U)"] \arrow[d, "\imath'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Sub(\mathcal{S}et) \arrow[d, "\imath"] \\ \omega\mathcal{C}po \arrow[r, "U"'] & \mathcal{S}et \end{tikzcd}$$

where U is the forgetful or global sections functor $\omega\mathcal{C}po(1, \_)$. As we have seen in §4.3.1, $\imath'$ is a fibred-ccc with $Cons_{\omega\mathcal{C}po}$-products. So we can make sense of logical predicates for $\omega\mathcal{C}po$ using classical logic, *i.e.* the internal logic of $\mathcal{S}et$.

There is a reflection $R \dashv J : Sub(\omega\mathcal{C}po) \to \kappa\mathcal{C}po$ (via $\eta, 1_{\kappa\mathcal{C}po}$), where J is the inclusion and R simply closes a subset $S \subseteq C$ (C a cpo) under sups of ω-chains.

More precisely

$$R(S) = \bigcap\{S' \text{ admissible subset of } C \mid S \subseteq S'\}$$

Note that the existence of a reflection at every fibre is guaranteed by Freyd's adjoint functor theorem [Mac71, p.116]; the existence of R then follows from Theorem 3 in [BGT91]. It is easy to verify that η satisfies the hypothesis of

Proposition 2.3.2 since the cpo structure on exponentials is given pointwise. Thus $Sub'(\omega\mathcal{C}po)$ is a ccc and we can interpret logical predicates in it.

The expression for exponents is then the same as for $\mathcal{S}et$:

$$(S \subseteq C) \Rightarrow (S' \subseteq D) \cong \{f : C \to D \text{ in } \omega\mathcal{C}po \mid \forall x : C.\ x \in S \Longrightarrow f\ x \in S'\}$$

which is an admissible subset of $C \Rightarrow D$.

4.3.1 Remarks.

- There are other ways of showing that $Sub'(\omega\mathcal{C}po)$ is a ccc, as in [MR91, MS92]. The method used here reflects better the logical nature of the constructions involved: since the logic of admissible subsets over $\omega\mathcal{C}po$ is not rich enough to interpret the fragment of predicate logic required to express logical exponents, we interpret them classically (that is, in $\mathcal{S}et$ and reflect them back into the above logic. So we are using the logic/fibration relation in an essential way.

- For the category $\omega\mathcal{C}po_\perp$, consisting of ω-complete posets with a bottom element and continuous functions, consider the functor $\imath : Sub'(\omega\mathcal{C}po_\perp) \to \omega\mathcal{C}po_\perp$, where the objects of $Sub'(\omega\mathcal{C}po_\perp)$ are admissible subsets, *i.e.* subsets closed under suprema of ω-chains and bottom element. The functor $\imath$ admits cartesian liftings along strict functions, via inverse images as before. We can thus infer the cartesian closed structure of $Sub'(\omega\mathcal{C}po_\perp)$ because the base morphisms involved in the construction of exponentials, namely projections and evaluation, are strict. So, the above construction can then be carried out in this setting. Anyway, logics for categories like $Sub'(\omega\mathcal{C}po_\perp)$ should be studied in the context of fibrations for categories of *partial maps*. See [KN93] for some preliminary considerations on this topic, using indexed categories.

- Note that the internal logics of $\omega\mathcal{C}po$ and $S\omega\mathcal{C}po_\perp$ allow any subset, with the discrete or flat ordering respectively, as a predicate. The restriction to admissible subsets is necessary if we want Scott's induction principle to reason about least fixed points.

4.3.3 Kipke logical predicates

Kripke lambda models were introduced in [MM87] to give Kripke-style semantics for the simply typed λ-calculus. Kripke models are complete for the usual proof system of simply typed λ-calculus, unlike Henkin models. By a *Kripke lambda model* here we mean a model of simply typed λ-calculus in the presheaf topos $\mathcal{S}et^{\mathcal{W}}$, with $\mathcal{W}$ the poset of 'possible worlds' regarded as a category in the usual way. Thus a Kripke lambda model is simply a sub-ccc of $\mathcal{S}et^{\mathcal{W}}$.

Consequently, Kripke logical predicates, as given in [Mit90, MM87] arise by carrying out the constructions of logical products and exponentials in the internal logic of $\mathcal{S}et^{\mathcal{W}}$, *i.e.* in the fibration $\begin{smallmatrix} Sub(\mathcal{S}et^{\mathcal{W}}) \\ \downarrow \imath \\ \mathcal{S}et^{\mathcal{W}} \end{smallmatrix}$ This fibration, like the internal logic of any topos, is a fibred-ccc with $Cons_{\mathcal{S}et^{\mathcal{W}}}$-products. For a description of the internal logic of toposes, and presheaf toposes in particular, see [LS86, Bel88, Gol79]. We only review those aspects relevant to the present application. Kripke logical predicates are used to show completeness results for a special kind of Kripke models in [MM87].

An object A in $\mathcal{S}et^{\mathcal{W}}$ consists of a W-indexed family of sets A^w and transition functions $A_{w,w'}$ for $w \leq w' \in W$, satisfying the evident composition and identity laws. The translation functors of the fibration perform substitution (regarding an object in the fibre $Sub(\mathcal{S}et^{\mathcal{W}})_A$ as a W-indexed family of subsets/predicates $P^w \subseteq A^w$):

$$((f : B \to A)^*(P \hookrightarrow A))^w = \{x \in B^w | P^w(f^w x)\}$$

The cartesian closed structure of the fibre $Sub(\mathcal{S}et^{\mathcal{W}})_A$ can be described as follows:

Terminal object: $1^w = A^w$

Binary products:

$$(P \times_A Q)^w = \{x \in A^w | P^w(x) \wedge Q^w(x)\}$$

Exponentials:

$$(P \Rightarrow_A Q)^w = \{x \in A^w | \forall w' \geq w. P^{w'}(A_{w,w'}x) \Longrightarrow Q^{w'}(A_{w,w'}, x)\}$$

The corresponding transition functions are induced by those of A, *e.g.* $(P \Rightarrow_A Q)_{w,w'} = A_{w,w'}$.

$Cons_{\mathcal{S}et^{\mathcal{W}}}$-products $\Pi_A : Sub(\mathcal{S}et^{\mathcal{W}})_{B\times A} \to Sub(\mathcal{S}et^{\mathcal{W}})_A$ are given by

$$(\Pi_A(P))^w = \{x \in B^w \mid \forall w' \geq w. \forall a \in A^{w'}. P^{w'}(B_{w,w'}x, a)\}$$

with transition functions induced by those of B.

The cartesian closed structure of $Sub(\mathcal{S}et^{\mathcal{W}})$ is given as follows:

Terminal object: $1^w(x) = \top$

Binary products: for $P \hookrightarrow A, Q \hookrightarrow B$

$$(P \times Q)^w = \{\langle x, y\rangle \in A^w \times B^w | P^w(x) \wedge Q^w(y)\}$$

Exponentials: for $P \hookrightarrow A, Q \hookrightarrow B$

$$\begin{aligned}
(P \Rightarrow Q)^w &= \{f \in (A \Rightarrow B)^w | \forall w' \geq w. \forall a \in A^{w'}. \\
&\quad (\forall w'' \geq w'. \\
&\quad P^{w''}(A_{w',w''}a) \Longrightarrow Q^{w''}(ev^{w''}_{A,B}\langle (A \Rightarrow B)_{w,wt''}f, \\
&\qquad\qquad\qquad\qquad A_{w',w''}a\rangle))\} \\
&= \{f \in (A \Rightarrow B)^w \mid \forall w' \geq w. \forall a \in A^{w'}. \\
&\quad P^{w'}(a) \Longrightarrow Q^{w'}(ev^{w'}_{A,B}\langle (A \Rightarrow B)_{w,w'}f, a\rangle)\}
\end{aligned}$$

This is precisely the definition of Kripke logical predicates in [MM87], where the authors notice that such predicates are obtained by interpreting the usual first-order formulas for logical predicates over applicative structures in the internal logic of a topos, although the connection to cartesian closure is not mentioned.

4.3.4 Deliverables

The examples presented so far have dealt with internal logics. Recalling the interpretation of first-order predicate calculus in a first-order hyperdoctrine, we see that for internal logics (*i.e.* subobject fibrations) there is at most one proof for a sequent. This aspect is called proof-irrelevance, since any two proofs of a derivable sequent are identified. However, in a 'term' hyperdoctrine, the fibres will not be preorders. The category of deliverables which we analyse next uses such a syntactic hyperdoctrine built from Coquand-Huet's Calculus of Constructions [CH88]. Actually, first-order intuitionistic predicate calculus over a simply typed λ-calculus as object language will suffice for the present example.

The category $\mathcal{D}el$ of first-order deliverables has

Objects A type s, together with a predicate S over $s, S : s \to \mathsf{Prop}$; a predicate over s is simply an s-indexed proposition.

Morphisms Pairs $(f, p) : (s, S) \to (t, T)$, such that $f : s \to t$ and $x : s \vdash p(x) : Sx \Longrightarrow T(fx)$. Such a pair is called a *deliverable*.

Identity $(1_s, (\lambda r : Sx.r)) : (s, S) \to (s, S)$

Composition For $(f, p) : (s, S) \to (t, T), (g, q) : (t, T) \to (u, U)$,

$$(g, q) \circ (f, p) = (g \circ f, (\lambda r : Sx.q(fx)(pr)))$$

This category was introduced in [BM91], as the basis of an approach to program development within the Calculus of Constructions. This approach integrates the classical 'Hoare-triples' assertions of partial correctness with the synthetic approach to programming which extracts a program from a constructive proof of its specification. The idea is that a morphism (f, p)

corresponds to a 'program' f and a 'proof of correctness' p w.r.t. to the input-output specification S-T. In *ibid.*, the authors show that the category of deliverables is cartesian closed, a fact which is exploited to structure program development. This assumes mild variations to the Calculus of Constructions, namely, the existence of unit types and η-conversion. Here, we show that cartesian closure is an immediate consequence of Corollary 3.3.11.

Let $\mathbb{B}$ be the (cartesian closed) category of types and terms of Calculus of Constructions, underlying the category of deliverables. Define the $\mathbb{B}$-indexed category $Del : \mathbb{B}^{op} \to \mathcal{C}at$ as follows: for a type $s, Del(s)$ has predicates $S : s \to \mathsf{Prop}$ as objects; a morphism $p : S \to T$ is a proof $x : s \vdash p(x) : Sx \Longrightarrow Tx$. For a morphism (term) $f : s \to t$, the reindexing functor $f^* : Del(t) \to Del(s)$ performs substitution:

$$\begin{aligned} f^*(T : t \to \mathsf{Prop}) &= \lambda x : s.T(fx) \\ f^*(p : S \to T) &= x : s \vdash p(fx) : S(fx) \Longrightarrow T(fx) \end{aligned}$$

The category $\mathcal{D}el$ is the fibred category over $\mathbb{B}$ obtained from Del via the Grothendieck construction. It is a fibred-ccc, with the evident 'pointwise' structure, *e.g.*

$$(S : s \to \mathsf{Prop}) \Rightarrow_s (S' : s \to \mathsf{Prop}) = \lambda x : s.Sx \Longrightarrow S'x$$

and it has $\mathrm{Cons}_{\mathbb{B}}$-products, given by Π-types:

$$\begin{aligned} \Pi_s(S : t \times s \to \mathsf{Prop}) &= \lambda y : t.\Pi x : s.S(x, y) \\ \Pi_s(p : S \to S') &= \lambda p' : \Pi x : s.S(x, y).\lambda x : s.p(x, y)(p'x) \end{aligned}$$

By Corollary 3.3.11, $\mathcal{D}el$ is cartesian closed, with cartesian closed structure given by logical predicates, essentially as in [BM91]. Note that a cartesian lifting for a 'program' $p : s \to t$ and a predicate $T : t \to \mathsf{Prop}$ yields a *weakest precondition* for p and T. Therefore, the notion of fibration is at the basis of the original work on the axiomatic approach to sequential program verification using Hoare-triples. The expression of such triples in a type-theoretic setting leads to the abovementioned deliverables. In a later version [BM92], the authors refined the structure of the category of deliverables to reflect more closely the Calculus of Constructions, which does not have η-conversion, in terms of semi-cartesian closed categories. Further, they introduced a category of second-order deliverables to allow the input and output of a program to be related in an specification. Such construction amounts to a polynomial fibration over $\mathcal{D}el$. The general construction of such polynomial fibrations is given in §6.

4.3.5 Categories derived from realisability

As a final example of a fibred-ccc with products, we take a brief look at categories defined in terms of Kleene's realisability interpretation of intuitionistic logic. Specifically, we show how the cartesian closed structure of $\omega - \mathcal{S}et$ and PER can be inferred from Corollary 3.3.11 and Propositions 2.3.1 and 2.3.2. The fact that these categories are cartesian closed, follows from topos-theoretic considerations. However, we include this material as an illustration; there is no claim of originality. The basic material concerning the above categories and their applications in the interpretation of polymorphic

λ-calculi can be found in [Pho92].

Let ω denote the set of natural numbers and $\wp\omega$ its powerset. For any set X, we define the following preorder on $X \Rightarrow \wp\omega$:

$$p \leq q \triangleq \exists r \in \omega.\forall x \in X, n \in \omega.n \in p(x) \Longrightarrow r \cdot n \downarrow \wedge r \cdot n \in q(x)$$

where $r \cdot n$ denotes Kleene's application and $\_ \downarrow$ is the 'definedness' predicate. Let $R : \mathcal{S}et^{op} \to \mathcal{C}at$ be the following $\mathcal{S}et$-indexed category:

$$\begin{array}{rcl} R(X) & = & (X \Rightarrow \wp\omega, \leq) \\ R(f : X \to Y) & = & \_ \circ f \end{array}$$

This indexed category is a so-called *tripos*. See [HJP80], where it is called the *recursive reahabdity tripos*. Consequently every $R(X)$ is a Heyting (pre-)algebra, with operations:

Top element: $\top_X : X \to \wp\omega \triangleq (x \in X \mapsto \omega)$

Binary meet: Let $\langle \_, \_ \rangle : \omega \times \omega \to \omega$ be a recursive pairing function. Then,

$$p \times_X q \triangleq (x \in X \mapsto \{\langle n, m \rangle \mid n \in p(x) \wedge m \in q(x)\})$$

Heyting implication: Let $A, B \subseteq \omega$ and

$$A \longrightarrow B \triangleq \{n \in \omega \mid \forall m \in \omega.m \in A \Longrightarrow n \cdot m \downarrow \wedge n \cdot m \in B\}$$

Then,

$$p \Rightarrow_X q \triangleq (x \in X \mapsto p(x) \longrightarrow q(x))$$

Let $\begin{array}{c}\mathcal{R}\\ \downarrow p\\ \mathcal{S}et\end{array}$ be the fibration associated to R. From the above, we already know that p is a fibred-ccc. It also has $\mathrm{Cons}_{\mathcal{S}et}$-indexed products: for $x, x' \in X$, let

$$\delta_X(x, x') = \begin{cases} \omega & \text{if } x = x' \\ \emptyset & \text{otherwise} \end{cases}$$

Then,

$$\Pi_X(p : Y \times X \to \omega)(y) \triangleq \{n \mid \forall (y', x) \in Y \times X . n \in (\delta_Y(y, y') \longrightarrow p(y', x))\}$$

By Corollary 3.3.11, $\mathcal{R}$ is a ccc. Denoting the objects of $\mathcal{R}$ as $A = \langle |A|, p_A \rangle$, the cartesian closed structure is given by

Terminal object: $\langle \{*\}, \top_{\{*\}} \rangle$

Binary products:

$$A \times B = \langle |A| \times |B|, p_{A\times B} \rangle$$

where

$$p_{A\times B}(a, b) = \{\langle n, m \rangle \mid n \in p_A(a) \wedge m \in p_B(b)\}$$

Exponentials:

$$A \Rightarrow C = \langle |A| \Rightarrow |B|, p_{A\Rightarrow B} \rangle$$

where

$$\begin{aligned} p_{A\Rightarrow B} = \{n| \quad & \forall a \in A . \forall m \in \omega n \cdot m \downarrow \wedge \\ & (\forall r \in \omega . r \in p_A(a) \Longrightarrow n \cdot m \cdot r \downarrow \wedge n \cdot m \cdot r \in p_B(f(a)))\} \end{aligned}$$

There are interesting subcategories of $\mathcal{R}$ with rich structure. One well-known example is the category of ω-sets [Pho92], which we now describe.

Given an object $A = \langle |A|, p_A \rangle$ of $\mathcal{R}$, an element $a \in |A|$ is said to be realisable iff $p_A(a) \neq \emptyset$. ω-$\mathcal{S}et$ is isomorphic to the full subcategory of $\mathcal{R}$ with objects $A = \langle |A|, p_A \rangle$ such that every $a \in |A|$ is realisable. Here is the more conventional scription of ω-$\mathcal{S}et$:

Objects: pairs $\langle |A|, \vdash_A \rangle$, where $|A|$ is a set and $\vdash_A \subseteq \omega \times |A|$ such that

$$\forall a \in |A|. \exists n \in \omega. n \vdash_A a$$

The pair $(|A|, \vdash_A)$ is an ω-set.

Morphisms: $f : \langle |A|, \vdash_A \rangle \to \langle |B|, \vdash_B \rangle$ is a function $f : |A| \to |B|$ in $\mathcal{S}et$ such that

$$\exists r \in \omega.\ \forall a \in |A|.\ \forall n \in \omega. n \vdash_A a \Longrightarrow r \cdot n \downarrow \wedge r \cdot n \vdash_B f(a)$$

As mentioned before ω-$\mathcal{S}et$ is cartesian closed:

Terminal object: $\langle \{*\}, \{(n, *) \mid n \in \omega\} \rangle$

Binary products:

$$\langle |A|, \vdash_A \rangle \times \langle |B|, \vdash_B \rangle = \langle |A| \times |B|, \vdash_{A \times B} \rangle$$

where

$$\langle n, m \rangle \vdash_{A \times B} \langle a, b \rangle \Leftrightarrow n \vdash_A a \wedge m \vdash_B b$$

Exponentials: $\langle |A|, \vdash_A\rangle \Rightarrow \langle |B|, \vdash_B\rangle = \langle \omega\text{-}\mathcal{S}et(\langle |A|, \vdash_A\rangle, \langle |B|, \vdash_B\rangle), \vdash_{A\Rightarrow B}\rangle$

where

$$n \vdash_{A\Rightarrow B} f \Leftrightarrow \forall a \in |A|.\forall m \in \omega. m \vdash_A a \Longrightarrow n \cdot m \downarrow \wedge n \cdot m \vdash_B f(a)$$

The finite products are as in $\mathcal{R}$, module the obvious change of notation $n \vdash_A a \leftrightarrow n \in p_A(a)$ that construes relations in $\omega \times |A|$ as functions $|A| \to \wp\omega$ and vice versa. But exponentials in ω-$\mathcal{S}et$ do not agree with those in $\mathcal{R}$. However, if we consider the realisable elements of the exponential of two ω-sets in $\mathcal{R}$ they correspond to the elements of the exponential in ω-$\mathcal{S}et$. This follows from Proposition 2.3.1 and the proposition below which sets up a suitable coreflection between ω-$\mathcal{S}et$ and $\mathcal{R}$.

We can obtain an ω-set from any object of $\mathcal{R}$ by discarding the non-realisable elements of the underlying set. This leads to the following

4.3.2. Proposition. *There is a coreflection*

$$\begin{tikzcd} \omega\text{-}\mathcal{S}et \arrow[r, shift right, "J"'] \arrow[r, phantom, "\top"] & \mathcal{R} \arrow[l, shift right, "Re"'] \end{tikzcd}$$

where $J : \omega\text{-}\mathcal{S}et \to \mathcal{R}$ is the full and faithful, finite-product preserving functor

$$\begin{aligned} J(\langle |A|, \vdash_A\rangle) &= \langle |A|, (a \in |A| \mapsto \{n \in \omega \mid n \vdash_A a\})\rangle \\ J(f) &= f \end{aligned}$$

Proof. Let $Re : \mathcal{R} \to \omega\text{-}\mathcal{S}et$ be the functor defined by

$$\begin{aligned} Re(\langle |A|, p_A \rangle) &= \langle \{a \mid p_A(a) \neq \emptyset\}, \{(n,a) \mid n \in p_A(a)\} \rangle \\ Re(f) &= f \end{aligned}$$

The corresponding hom-set isomorphism

$$\theta_{A,B} : \mathcal{R}(J(\langle |A|, \vdash_A \rangle), \langle |B|, p_B \rangle) \xrightarrow{\sim} \omega\text{-}\mathcal{S}et(\langle |A|, \vdash_A \rangle, Re(\langle |B|, p_B \rangle))$$

is given by

$$\theta_{A,B}(f : J(\langle |A|, \vdash_A \rangle) \to \langle |B|, p_B \rangle) = \hat{f} : |A| \to \{b \in |B| \mid p_B(b) \neq \emptyset\}$$

where $\hat{f}(a) = f(a)$ with the same realiser as f; f must take values in $\{b \in |B| \mid p_B(b) \neq \emptyset\}$ since otherwise it will not be realisable. Conversely,

$$\theta^{-1}_{A,B}(g : \langle |A|, \vdash_A \rangle \to Re(\langle |B|, p_B \rangle)) = \iota \circ g$$

with the same realiser as g and $\iota : \{b \in |B| \mid p_B(b) \neq \emptyset\} \hookrightarrow |B|$ is the inclusion. It is clear that the non-realisable objects of $|B|$ have no effect in the realisability of functions with codomain $|B|$. Naturality of $\theta_{A,B}$ is immediate. □

Note that applying Corollary 2.3.1, exponentials in $\omega\text{-}\mathcal{S}et$ are given by

$$\langle |A|, \vdash_A \rangle \Rightarrow_{\omega\text{-}\mathcal{S}et} \langle |B|, \vdash_B \rangle \cong Re(J\langle |A|, \vdash_A \rangle) \Rightarrow_{\mathcal{R}} J(\langle |B|, \vdash_B \rangle))$$

which agree with the previously given description.

Another interesting category arising from realisability is PER [Pho92]. Its objects are the symmetric and transitive relations on ω. For R, S two such relations, a morphism $f : R \to S$ in PER is a function $f : Q(R) \to Q(S)$, where $Q(R) = \{[n]_R \mid n \in dom(R)\}$ is the set of equivalence classes of R, such that there is a realiser $n \in \omega$ satisfying

$$\forall m \in dom(R). f[m]_R = [n \cdot m]_R$$

We say that n *realises* f. PER is equivalent to the category $\mathcal{M}od$ of so-called "modest ω-sets", *ibid.* This is the full subcategory of ω-$\mathcal{S}et$ with objects $\langle |A|, \vdash_A \rangle$ satisfying

$$\forall a, a' \in |A|. \forall n \in \omega. (n \vdash_A a \wedge n \vdash_A a') \Longrightarrow a = a'$$

The equivalence with PER is given by

$$\mathcal{M}od \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathsf{PER}$$

with

$$\begin{array}{rcl} \Phi\langle |A|, \vdash_A \rangle & = & \{(n, m) \mid \exists a \in |A|. n \vdash_A a \wedge m \vdash_A a\} \\ \Psi(R) & = & \langle Q(R), \in \rangle \end{array}$$

PER is cartesian closed. Exponentials are given as follows: for R, S in PER

$$R \Rightarrow^{S}_{\mathsf{PER}} = \{(n, n') \mid \exists f : R \to S. n \text{ realises } f \wedge n' \text{ realises } f\}$$

We can infer this from the fact that $\mathcal{M}od$ is a reflective subcategory of ω-$\mathcal{S}et$, applying Proposition 2.3.2. The reflection $\Theta : \omega\text{-}\mathcal{S}et \to \mathcal{M}od$ is given as follows: for an ω-set $\langle |A|, \vdash_A \rangle$, define a relation $\smile$ on $|A|$ by

$$a \smile a' \triangleq \exists n \in \omega . n \vdash_A a \wedge n \vdash_A a'$$

Let $\sim$ denote the transitive closure of $\smile$. Then $\Theta A = \langle |A| \backslash_\sim, \vdash_{\Theta A} \rangle$, with

$$n \vdash_{\Theta A} [a]_\sim \triangleq \exists a' \in [a]_\sim . n \vdash_A a'$$

To verify that this reflection satisfies the hypothesis of Proposition 2.3.2, observe that its unit is $\eta_A = [\_]_\sim : \langle |A|, \vdash_A \rangle \to \langle |A| \backslash_\sim, \vdash_{\Theta A} \rangle$, realised by $\lceil I \rceil$, a code for the identity function. Any modest set is isomorphic to ΨR for some $R \in |\mathsf{PER}|$ because of the abovementioned equivalence. Therefore the exponential of two modest sets in ω-$\mathcal{S}et$ is

$$\langle Q(R), \in \rangle \Rightarrow \langle Q(S), \in \rangle = \langle \{ f : Q(R) \to Q(S) \mid f \text{ in } \omega\text{-}\mathcal{S}et \}, \vdash_{\Psi R \Rightarrow \Psi S} \rangle$$

In the modest set $\Theta(\langle Q(R), \in \rangle) \Rightarrow \langle Q(S), \in \rangle)$, we must identify the f's in (the transitive closure of) the relation

$$f \smile f' \equiv \exists r \in \omega . r \text{ realises } f \wedge r \text{ realises } f'$$

But then the definition of 'r realises f' implies that

$$f \smile f \equiv f = f'$$

Therefore, the function part of the unit $\eta_{\Psi R \Rightarrow \Psi S}$ is $f \mapsto \{f\}$, which is an isomorphism. It is easy to verify that

$$R \Rightarrow_{\mathsf{PER}} S = \Phi\Theta(\Psi(R) \Rightarrow_{\omega\text{-}\mathcal{S}et} \Psi(S))$$

We have thus shown how Corollary 3.3.11 and the fairly general properties of reflections and coreflections given in Propositions 2.3.1 and 2.3.2 allows us to infer the cartesian closed structure of $\mathcal{R}, \omega$-$\mathcal{S}et$ and PER.

4.4 A related categorical approach to logical predicates

A categorical approach to logical predicates has been proposed in [MR91]. It seems appropriate to make a few comparisons between this approach and the one we have presented in §4.2. In *ibid.* a category of relations is defined to study parametricity issues arising in first and second order lambda calculus. For this purpose, the authors define for given categories $\mathbb{K}$ and $\mathbb{B}$ and a functor $F : \mathbb{K} \to \mathbb{B}$, the category of relations over $\mathbb{K}$, $\mathcal{R}el(\mathbb{K}, \mathbb{B}, F)$, as follows:

Objects: $\langle K, B, m \rangle \in |\mathcal{R}EL(\mathbb{K}, \mathbb{B}, F)\ |$ iff $K \in |\mathbb{K}|, B \in |\mathbb{B}|$ and $m : B \hookrightarrow FK$ monic

Morphisms:

$$\langle f, g \rangle : \langle K, B, m \rangle \to \langle K', B', m' \rangle \text{ iff } f : K \to K', g : B \to B'$$

and

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, hook, "m"'] & B' \arrow[d, hook, "m'"] \\ FK \arrow[r, "Ff"] & FK' \end{tikzcd}$$

commutes in $\mathbb{B}$.

Composition and identities are defined componentwise. There is a forgetful functor $U : \mathcal{R}el(\mathbb{K}, \mathbb{B}, F) \to \mathbb{K}$ such that

$$U\langle K, B, m\rangle = K \qquad U\langle f, g\rangle = f$$

This category, which is intended as a direct generalisation of the category of set-theoretic predicates $\mathcal{R}el(\mathcal{S}et, \mathcal{S}et, 1_{\mathcal{S}et})$, can easily be expressed in terms of fibrations. From the subobject fibration $\imath : Sub(\mathbb{B}) \to \mathbb{B}$ we obtain by change of base along F $\mathcal{R}el(\mathbb{K}, \mathbb{B}, F) \cong F^*(Sub(\mathbb{B}))$, and the functor U is the projection from the pullback to $\mathbb{K}$. This definition makes sense regardless of whether $\imath$ is a fibration; in case it is, $\mathcal{R}el(\mathbb{K}, \mathbb{B}, F)$ is fibred over $\mathbb{K}$, via U.

In [MR91], Ma and Reynolds go on to analyse some properties of $\mathcal{R}el(\mathbb{K}, \mathbb{B}, F)$. They show that if $\mathbb{K}$ is a ccc, $\mathbb{B}$ is a ccc with finite limits and F preserves finite products then $\mathcal{R}el(\mathbb{K}, \mathbb{B}, F)$ is a ccc and the projection functor into $\mathbb{K}$ preserves this structure. Such proposition is in the same spirit as our Corollary 3.3.11, although the hypotheses are different. However *ibid.* does not provide an explicit connection between such property of $\mathcal{R}el(\mathbb{K}, \mathbb{B}, F)$ and logical predicates, although in some particular cases such a relationship exists, *e,g.* when $\mathbb{B}$ is $\mathcal{S}et$, or any topos, the expression of the construction in *ibid.* in the internal language of U yields logical predicates.

[MR91] continues with the statement of the 'Identity Extension Lemma'. For this purpose, the authors define the functor $J : \mathbb{K} \to \mathcal{R}el(\mathbb{K}, \mathbb{B}, F)$ as follows:

$$\begin{array}{rcl} J(K) & = & \langle K, FK, 1_{FK} \rangle \\ J(f : K \to K') & = & \langle f, Ff \rangle \end{array}$$

That is, J takes an object to the identity relation over it, via F. The Identity Extension Lemma asserts that, under certain hypotheses, J is a cc-functor, *i.e.* preserves the cartesian closed structure.

The unary case of this lemma is immediate in our framework, since in this case J amounts to the fibred terminal object functor. This functor yields then a full cc-embedding of the base category in the 'category of predicates'. The case of binary relations requires the existence of equality predicates in the fibration $\begin{smallmatrix} \mathcal{R}el(\mathbb{K}, \mathbb{B}, F) \\ \downarrow U \\ \mathbb{K} \end{smallmatrix}$ to define J. Equality predicates for fibrations have been characterised in [Law70]. For a fibration $\begin{smallmatrix} \mathbb{E} \\ \downarrow p \\ \mathbb{B} \end{smallmatrix}$, regarded as a logic over $\mathbb{B}$, the *equality predicate* on a type $A \in |\mathbb{B}|$ is given by the coreindexing, or direct image, $\delta_!(1_A)$, where $\delta : A \to A \times A$ is the diagonal morphism and 1_A is the terminal object over A, *cf.* Remark 4.2.1. This means that the equality predicate on A is characterised as the least reflexive relation on A. A similar approach applies to n-ary relations. Thus, we can express the Identity Extension Lemma for p. When formulated in the internal language of p, it amounts to the requirement that the equality predicate on an exponential type $A \Rightarrow B$ be given pointwise:

$$f, g : A \longrightarrow B \vdash f =_{A \longrightarrow B} g \leftrightarrow \forall x, y : A.x =_A y \Longrightarrow \mathrm{ev}\langle f, x \rangle =_B \mathrm{ev}\langle g, y \rangle$$

Ma and Reynolds extend their analysis of categories of relations to deal with logical relations for system F in terms of PL-categories [See87], although they make no explicit connection between the categorical constructions they present and second-order logical relations. To give an abstract account of these further research is required. We comment on a possible direction to follow in §7.

4.5 Induction principle for data types in a fibration

We have seen in §4.2 that for $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ the expression of certain structure of $\mathbb{E}$ in the internal language of p allows us to obtain certain logical concepts, namely logical predicates, from categorical ones, namely cartesian closure. We have also seen in §4.4 how this 'logical structure' of $\mathbb{E}$ can be used to assert the validity of certain logical principles, like pointwise equality for $\longrightarrow$-types, by requiring certain functors to preserve such structure. In this section we give another instance of this, providing a categorical interpretation of structural induction for data types.

Following [CS91, Jac93], we will consider inductive data types in a distributive category $\mathbb{B}$. We only review the concepts required to formulate the abovementioned induction principles for such types. The material on distributive categories and inductive datatypes in them is taken from [Jac93].

4.5.1. Definition.

- A category $\mathbb{B}$ with finite products and finite coproducts is *distributive* if, for every $I \in |\mathbb{B}|$, the functor $I \times \_ : \mathbb{B} \to \mathbb{B}$ preserves finite coproducts.
- A functor $F : \mathbb{B} \to \mathbb{C}$ between distributive categories $\mathbb{B}$ and $\mathbb{C}$ is *distributive* if it preserves finite products and coproducts.

4.5.2. Example. Any cartesian closed category $\mathbb{C}$ with finite coproducts is distributive: for any $I \in |C|, I \times \_$ preserves coproducts because it has a right adjoint. Thus, $\mathcal{S}et, \omega\mathcal{C}po$ and PER are distributive categories.

Inductive data types in a distributive category are specified by means of endofunctors, which give the signature of the type. To formulate a precise definition of models for such specifications, we consider the category of algebras for an endofunctor:

4.5.3. Definition. Given a functor $T : \mathbb{B} \to \mathbb{B}$, the category T-$\mathcal{A}lg$ has:

Objects: pairs $(X, x : TX \to X)$, called T-algebras.

Morphisms: $f : (X, x) \to (Y, y)$ is a morphism $f : X \to Y$ in $\mathbb{B}$, such that $f \circ x = y \circ Tf$.

Composition and identities are inherited from $\mathbb{B}$.

4.5.4. Definition. Let $\mathbb{B}$ be a distributive category, S a finite set and $M : S \to \mathbb{B}$ be a functor, regarding S as a discrete category.

(i) Let $\mathcal{T}_M \subseteq |\mathcal{C}at(\mathbb{B}, \mathbb{B}|$ be the least class of endofunctors on $\mathbb{B}$ such that

- The identity functor is in $\mathcal{T}$.
- For any $I \in S$, the constant functor $X \mapsto I$ is in $\mathcal{T}$.
- If T_1 and T_2 are in $\mathcal{T}$, so are $T_2 \circ T_1$, $T_1 \times T_2$ and $T_1 + T_2$.

(ii) An *inductive data type specification* in $\mathbb{B}$, idts for short, is given by a functor $M : S \to \mathbb{B}$ and a functor $(T : \mathbb{B} \to \mathbb{B}) \in \mathcal{T}_M$. We write T_M for such idts.

(iii) An *model* for an idts T_M is a T-algebra.

(iv) The *initial model* for an idts T_M is the initial T-algebra (if it exists).

The set S in the above definition is called a *parameter set*. Its role is to specify, via the functor $M : S \to \mathbb{B}$, those objects of $\mathbb{B}$ which are parameters for the data type specified. The examples below will make this clear. See [Jac93] for a more general and elegant formulation of data types in distributive categories. The initial T-algebra of a functor $T : \mathbb{B} \to \mathbb{B}$ need not exist. But it is possible to guarantee the existence of initial T-algebras under suitable cocompleteness conditions on $\mathbb{B}$ and T. As shown in [LS81], an initial T-algebra can be obtained as the colimit of an ω-chain, when T preserves such colimits. An ω-chain is a functor $\omega \to \mathbb{B}$, where ω is the poset category of natural numbers with their usual ordering. The initial T-algebra is the colimit of the following ω-chain:

$$0 \xrightarrow{\iota} T0 \xrightarrow{T\iota} T^2 0 \cdots$$

where $\iota : 0 \to T0$ is the unique morphism from the initial object. For $\mathbb{B} = \mathcal{S}et$, and $T \in \mathcal{T}_{-}$ preserves colimits of ω-chains and therefore any idts in $\mathcal{S}et$ has an initial model.

An important observation, due to Lambek [LS81], is that for an initial T- algebra $(D, \mathsf{constr} : TD \to D)$, constr is an isomorphism. Thus, we can regard D as the 'least fixed point' of T, as illustrated by the above ω-chain. The isomorphism constr provides the 'constructors' of the data type, as the following familiar examples illustrate.

4.5.5. EXAMPLES. Let $\mathbb{B}$ be a distributive category.

(i) **Natural numbers object:** Consider the idts $TX = 1 + X$, with parameter set $\emptyset$. A T-algebra $(A, [c, f] : TA \to A)$ is given by an object A, the 'carrier' of the type, and morphisms $c : 1 \to A$ and $f : A \to A$. An initial model for T is precisely a natural numbers object $(N, [z, s])$ in Lawvere's sense, see [LS86, Part I,§9]. In $\mathcal{S}et$, it is the set of natural numbers ω, with the usual 0 and successor operations. Initiality means that there is an 'iterator', which given c and f as above produces a morphism $h : N \to A$ such that $h \circ z = c$ and $h \circ s = f \circ h$. In $\mathcal{S}et$, h corresponds to the function defined from c and f by primitive recursion. We write $it(c, f)$ for h above.

(ii) **Lists:** For an object $A \in |\mathbb{B}|$, consider the idts $T_A X = 1 + A \times X$, for a singleton parameter set, *i.e.* $A : \{*\} \to \mathbb{B}$. A T-algebra is given by an object B and morphisms $c : 1 \to B$ and $t : A \times B \to B$. An initial model in $\mathcal{S}et$ is precisely the set $List(A)$ of finite lists of elements of A, with the usual operations $nil : 1 \to List(A)$, the empty list, and $cons : A \times List(A) \to List(A)$, which given $a \in A$ and a list l, returns this list with the element a appended to its head.

The example of lists above shows the role of the parameter S and the functor $M : S \to \mathbb{B}$ in the specification of a data type; the type of lists $List(A)$ is parameterised by the type A of the elements of the list.

Consider now $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$, with $\mathbb{B}$ a distributive category. We will use Corollaries 3.3.6 and 3.3.7 to impose sufficient conditions on p to make $\mathbb{E}$ a distributive category. We will then consider the idts on $\mathbb{E}$ induced by a given idts on $\mathbb{B}$ to assert an induction principle for the latter. We will need the following Frobenius condition [Law70] on coreindexing functors for p:

4.5.6. Definition. Let $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ be a fibration with fibred binary products, and let $u : I \to J$ be a morphism in $\mathbb{B}$ for which a coreindexing functor, given by cocartesian liftings, $u_! : \mathbb{E}_I \to \mathbb{E}_J$ exists. $u_!$ satisfies *Frobenius* if, for every $X \in |E_J|$ and every $Y \in |E_I|$, the canonical morphism

$$\langle \epsilon_X \circ u_!\pi, u_!\pi' \rangle : u_!(u^*(X) \times Y) \to X \times u_!(Y)$$

is an isomorphism, where $\epsilon : u_! u^* \dot{\to} 1_{\mathbb{E}_J}$ is the counit of $u_! \dashv u^*$, *cf.* Proposition 1.2.7.

4.5.7. Remark. When p is a fibred-ccc, coreindexing functors for p satisfy Frobenius [Pit91].

4.5.8. PROPOSITION. *Given* $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{array}$ *with*

- $\mathbb{B}$ *a distributive category,*
- p *a fibred distributive category, i.e. every fibre is a distributive category and reindexing functors are distributive,*
- p *has coreindexing functors along coproduct injections,* $I \xrightarrow{\iota} I + J \xleftarrow{\iota'} J$, *for every* $I, J \in |\mathbb{B}|$. *Such coreindexing functors satisfy Frobenius and Beck-Chevalle condition.*

Then, $\mathbb{E}$ *is a distributive category and* p *strictly preserves finite products and coproducts.*

Proof. By Corollaries 3.3.6 and 3.3.7, $\mathbb{E}$ has finite products, $\tilde{\times}$ and $\tilde{1}$, and finite coproducts, $\tilde{+}$ and $\tilde{0}$, and p strictly preserves them. It only remains to verify that for any $X \in |\mathbb{E}|$, $X \tilde{\times}_{-} : \mathbb{E} \to \mathbb{E}$ preserves finite coproducts: given $Y \in |\mathbb{E}_J|$ and $Z \in |E_K|$, let $pX = I$ and $\zeta : (I \times J) + (I \times K) \xrightarrow{\sim} I \times (J + K)$ be the canonical isomorphism. Also, let

$$I \xrightarrow{\iota} J + K \xleftarrow{\iota'} K$$

and

$$I \times J \xrightarrow{\kappa} (I \times J) + (I \times K) \xleftarrow{\iota'} I \times K$$

be the corresponding coproduct diagrams. Note that

$$I \times \iota = \zeta \circ \kappa \qquad I \times \iota' = \zeta \circ \kappa'$$

by distributivity of $\mathbb{B}$. Then,

$$
\begin{array}{ll}
 & X \tilde{\times} (Y \tilde{+} Z) \\
\cong & \pi^*_{I,J+Z}(X) \times_{I\times(J+K)} (\pi'_{I,J+Z})^*(\iota_!(Y) +_{J+K} \iota'_!(Z)) \\
\cong & \pi^*_{I,J+Z}(X) \times_{I\times(J+K)} (\pi'_{I,J+Z})^*(\iota_!(Y)) \\
 & \quad +_{I\times(J+K)} (\pi^*_{I,J+Z}(X) \times_{I\times(J+K)} (\pi'_{I,J+Z})^*\iota'_!(Z) \\
 & \quad \text{by fibred distributivity} \\
\cong & \pi^*_{I,J+Z}(X) \times_{I\times(J+K)} (I \times \iota)_!(\pi'_{I,J})^*(Y)) \\
 & \quad +_{I\times(J+K)} (\pi^*_{I,J+Z}(X) \times_{I\times(J+K)} (I \times \iota')_!(\pi'_{I,K})^*(Z)) \\
 & \quad \text{by Beck-Chevalley condition} \\
\cong & ([\pi_{I,J}, \pi_{I,K}]^*(X) \times_{I\times J} \kappa_!(\pi'_{I,J})^*(Y)) \\
 & \quad +_{(I\times J)+(I\times K)} ([\pi_{I,J}, \pi_{I,K}]^*(X) \times_{I\times K} \kappa'_!(\pi'_{I,K})^*(Z)) \\
 & \quad \text{by reindexing along } \zeta \\
\cong & \kappa_!(\pi^*_{I,J}(X) \times_{I\times J} (\pi'_{I,J})^*(Y)) +_{(I\times J)+(I\times K)} \kappa'_!(\pi^*_{I,J}(X) \times_{I\times K} (\pi'_{I,K})^*(Z)) \\
 & \quad \text{by Frobenius} \\
\cong & (X \tilde{\times} Y) \tilde{+} (X \tilde{\times} Z)
\end{array}
$$

□

4.5.9. Remark. The Beck-Chevalley condition required for coreindexing functors in the above proposition implies that for a coproduct injection $\iota : J \to J + K$ and objects $I \in |B|$ and $X \in |\mathbb{E}_J|$, $(\pi'_{I,J+K})^*(\iota_!(X)) \cong (I \times \iota)_!((\pi^*_{I,J}(X))$. This is an instance of the Beck-Chevalley condition over the pullback square

$$
\begin{tikzcd}
I \times J \arrow[r, "\pi'"] \arrow[d, "I \times \iota"] & J \arrow[d, "\iota"'] \\
I \times (J + K) \arrow[r, "\pi'"'] & J + K
\end{tikzcd}
$$

See [Law70, Pav90] for further details.

4.5.10. EXAMPLE. The internal logic fibration $\begin{smallmatrix} Sub(\mathcal{S}et) \\ \downarrow\iota \\ \mathcal{S}et \end{smallmatrix}$ satisfies the hypotheses of Proposition 4.5.8. Hence $Sub(\mathcal{S}et)$ is a distributive category.

For $\begin{smallmatrix} \mathbb{E} \\ \downarrow p \\ \mathbb{B} \end{smallmatrix}$ satisfying the hypotheses of Proposition 4.5.8, given a set of parameters S and functors $M : S \to \mathbb{B}$ and $\tilde{M} : S \to \mathbb{E}$ such that $p\tilde{M} = M$, an idts $T_M : \mathbb{B} \to \mathbb{B}$ induces an idts $\tilde{T}_{\tilde{M}} : \mathbb{E} \to \mathbb{E}$ fibred over T, using the distributive structure of $\mathbb{E}$. The formal definition of $\tilde{T}_{\tilde{M}}$ proceeds by induction on the construction of $T \in \mathcal{T}_M$. For instance, given $H \in |\mathbb{E}_A|$, $T_A X = 1 + A \times X$ induces $\tilde{T}Y = \tilde{1} \tilde{+} H \tilde{\times} Y$. We can then consider $\tilde{T}$-algebras and initial models in $\mathbb{E}$.

4.5.11. LEMMA. *Given* $\begin{smallmatrix} \mathbb{E} \\ \downarrow p \\ \mathbb{B} \end{smallmatrix}$ *and a fibred 1-cell* $(\tilde{T}, T) : p \to p$, $\tilde{T}$-$\mathcal{A}lg$ *is added over* T-$\mathcal{A}lg$ *via the functor* p-$\mathcal{A}lg : \tilde{T}$-$\mathcal{A}lg \to T$-$\mathcal{A}lg$, *with action* $(X, x) \mapsto (pX, px)$.

Proof. Given (X, x) in $\tilde{T}$-$\mathcal{A}lg$ and $u : (J, j) \to pX,\ px$, a cartesian lifting for u is given as indicated by the following diagram

$$\begin{tikzcd}
\tilde{T}u^*(X) \arrow[r, "\tilde{T}\overline{u}(X)"] \arrow[d, dashed, "\overline{u}(x)"'] & \tilde{T}X \arrow[d, "x"] \\
u^*(X) \arrow[r, "\overline{u}(X)"'] & X
\end{tikzcd}$$

where the dashed morphism above is the unique morphism making the diagram commute with $p\overline{u}(x) = j$. $\square$

In the spirit of Definition 4.2.2, given $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ satisfying the hypotheses of Proposition 4.5.8, we could consider 'logical predicates' for a T-algebra (A, a) to be those $\tilde{T}$-algebras $(\tilde{A}, \tilde{a})$ over (A, a). Note that when T involves constant functors, given by an object $I \in |\mathbb{B}|$ say, a choice of an object H over I for the corresponding $\tilde{T}$-algebra corresponds logically, via the internal language of p, to a predicate over I.

When a fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ has a fibred terminal object $1 : \mathbb{B} \to \mathbb{E}$, it induces a functor $1\text{-}\mathcal{A}lg : T\text{-}\mathcal{A}lg \to \tilde{T}\text{-}\mathcal{A}lg$, for $(\tilde{T}, T) : p \to p$, by $(A, a) \mapsto (1(A), \overline{a}(1(A)) \circ !_{\tilde{T}1(A)})$, using the fact that $a^*(1(A))$ is terminal in $\mathbb{E}_{TA}$, and therefore there is a unique morphism $!_{\tilde{T}1(A)} : \tilde{T}1(A) \to a^*(1(A))$. We will use the functor $1\text{-}\mathcal{A}lg$ to relate initial models in $\mathbb{E}$ and $\mathbb{B}$ in the following proposition, and to formulate the induction principle in Definition 4.5.13.

4.5.12 Proposition. *Let* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *have a fibred terminal object* $1 : \mathbb{B} \to \mathbb{E}$ *and let* $(\tilde{T}, T) : p \to p$ *be a fibred 1-cell. If* $(\tilde{D}, \tilde{d})$ *is an initial* $\tilde{T}$*-algebra,*

$(p\tilde{D}, p\tilde{d})$ *is an initial T-algebra.*

Proof. Let $(D,d) = (p\tilde{D}, p\tilde{d})$. Given a T-algebra (A,a), we get a $\tilde{T}$-algebra $1\text{-}\mathcal{A}lg(A,a) = (1(A), \overline{a}(1(A)) \circ !_{\tilde{T}1(A)})$, as noted above. Hence, there is a unique morphism $h : (\tilde{D}, \tilde{d}) \to 1\text{-}\mathcal{A}lg(A,a)$, which induces a morphism $ph : (D,d) \to (A,a)$ of T-algebras. Given any other morphism $u : (D,d) \to (A,a)$, it induces a morphism $l(u) \circ !_{\tilde{D}} : (\tilde{D}, \tilde{d}) \to 1\text{-}\mathcal{A}lg(A,a)$. Thus $1(u) \circ !_{\tilde{D}} = h$ by initiality of $(\tilde{D}, \tilde{d})$ and so $u = ph$, which shows (D,d) is initial. $\square$

Thus, given the data in the above proposition and an initial T-algebra (D,d) we may look for an initial $\tilde{T}$-algebra over it. For $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ as in Proposition 4.5.8, given a parameter set S, a functor $M : S \to \mathbb{B}$ induces a functor $1M : S \to \mathbb{E}$, with $p1M = M$, via the terminal object functor $1 : \mathbb{B} \to \mathbb{E}$. Hence an idts $T_M : \mathbb{B} \to \mathbb{B}$ induces an idts $\overline{T} = \tilde{T}_{1M} : \mathbb{E} \to \mathbb{E}$. We can now express what it means for $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ regarded as a logic over $\mathbb{B}$, to satisfy an induction principle for an idts T_M in terms of the induced idts $\overline{T}$.

4.5.13. Definition. Let $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ satisfy the hypotheses of Proposition 4.5.8, and let $T_M : \mathbb{B} \to \mathbb{B}$ be an idts, for a parameter set S and a functor $M : S \to \mathbb{B}$. $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *satisfies the induction principle w.r.t.* T if $1\text{-}\mathcal{A}lg : T\text{-}\mathcal{A}lg \to \overline{T}\text{-}\mathcal{A}lg$ preserves initial models, *i.e.* whenever (D, constr) is an initial T-model, $1\text{-}\mathcal{A}lg(D, \mathsf{constr})$ is an initial $\tilde{T}$-model.

This definition means that for an object H in $\mathbb{E}$, to give a global element

$p : 1(D) \to H$ amounts to giving a $\tilde{T}$-algebra on H, $(H, h : \tilde{T}H \to H)$. We illustrate the logical import of the above definition with the idts of natural numbers and lists below. The internal language of p in this case includes the logical connectives $\{\wedge, \top, \vee, \bot\}$ and the coreindexing functors along coproduct injections. To simplify the presentation, we consider only the entailment relation $\vdash$ in the internal language, disregarding the proof terms. Note that for $\iota : I \to I + J$ in $\mathbb{B}$, given predicates $Q \triangleq x : I \vdash Q(x)\mathsf{Prop}$ and $P \triangleq y : I + J \vdash P(x)\mathsf{Prop}$ a morphism $f : \iota_! Q \to P$ corresponds under the adjunction $\iota_! \dashv \iota^*$ to a morphism $f^\wedge : Q \to \iota^*(P)$, which amounts to an entailment $x : I | Q(x) \vdash P(\iota x)$.

4.5.14. Examples. Let $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ be as in Proposition 4.5.8.

(i) For the idts $TX = 1 + X$ in $\mathbb{B}$, the corresponding $\overline{T}$ idts in $\mathbb{E}$ is $\overline{T}H = \tilde{1}\tilde{+}H$. Let $P \in |\mathbb{E}_I|$ and let $(N, [z, s])$ be the initial T-model in $\mathbb{B}$. To give a global element of P, we must give a $\tilde{T}$-algebra $(P, f : \tilde{T}P \to P)$. This amounts to giving a T-algebra $(I, [a, m] : TI \to I)$ — which induces a morphism $it(a, m) : N \to I$— and a vertical morphism $\hat{f} : \overline{T}P \to [a, m]^*(P)$. Let us examine this vertical morphism in the internal language of p: it amounts to a sequent

$$x : 1 + I \mid \iota_!(\top) \vee \iota'_!(P) \vdash P([a, m]x)$$

which can be decomposed into two sequents

$$x : 1 + I \mid \iota_!(\top) \vdash P([a, m]x) \quad x : 1 + I \mid \iota'_!(P) \vdash P([a, m]x)$$

which in turn correspond to sequents

$$x' : 1 \mid \top \vdash P(a) \qquad y : I \mid P(y) \vdash P(my)$$

which corresponds to the usual induction principle on the natural numbers: to prove $P(x)$ for the elements $x : I$ generated by a and m, we must prove $P(a)$ and $P(y) \Longrightarrow P(my)$.

(ii) For the idts $T_A X = 1 + A \times X$, for some $A \in |\mathbb{B}|$, we get the idts $\overline{T}Y = \tilde{1}\tilde{+}1(A)\tilde{\times}Y$. Let $(L[nil, cons])$ be the initial T-model and let $P \in |\mathbb{E}_L|$. Note that modulo the isomorphism $[nil, cons] : 1 + A \times L \to L$, the predicate P corresponds to a predicate P' on $1 + A \times L$, *i.e.* $x : 1 + A \times L \vdash P'(x)$. P' is therefore given by two predicates S and Q, with $x' : 1 \vdash S \leftrightarrow P'(nil)$ and $a : A, l : L \vdash Q(a, l) \leftrightarrow P'(cons(a, l))$. To give a vertical global element $h : l(L) \to P$, a proof of the property P for all lists, amounts to giving a morphism $k : \tilde{T}P \to P$ over $[nil, cons] : 1 + A \times L \to L$. It corresponds to a sequent

$$l : 1 + A \times L \mid \iota_!(\top) \vee \iota'_!(\top \times P) \vdash P'([nil, cons]l)$$

which can be decomposed into two sequents

$$x : 1 \mid \top \vdash S$$

and

$$a : A, l' : L \mid P(l') \vdash Q(a, l')$$

where we have simplified the antecedent of the second sequent by $\top \wedge P(l') \equiv P(l')$. We thus get the usual structural induction principle for finite lists.

We have thus given the categorical counterpart of the logical principle of structural induction by requiring the functor $1 : \mathbb{B} \to \mathbb{E}$ to preserve a suitable categorical property, *i.e.* initiality of algebras. Note that this is only possible if we consider the 'global' structure of the fibred category $\mathbb{E}$ rather than its fibred structure. This illustrates the value of working with fibrations rather than indexed categories.

It follows from [LS81, §5.2,Theorem 1] that when $\begin{array}{c}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{array}$ is the internal logic fibration $\begin{array}{c}Sub\mathbb{B}\\ \downarrow \iota\\ \mathbb{B}\end{array}$ logic fibration $\begin{array}{c}Sub\mathbb{B}\\ \downarrow \iota\\ \mathbb{B}\end{array}$ it satisfies the induction principle for any idts. We illustrate the argument for the case of natural numbers object.

4.5.15 PROPOSITION. *Given a category $\mathbb{B}$ with pullbacks and a natural numbers object $(N, [z, s])$, $\begin{array}{c}Sub\mathbb{B}\\ \downarrow \iota\\ \mathbb{B}\end{array}$ satisfies the induction principle for natural numbers.*

Proof. Recall that the idts for natural numbers is $TX = 1 + X$. For ι, an $\overline{T}$-algebra

$$\begin{tikzcd}
1 \arrow[r, "\tilde{c}"] \arrow[d] & P \arrow[r, "\tilde{f}"] \arrow[d, hook] & P \arrow[d, hook] \\
1 \arrow[r, "c"'] & A \arrow[r, "f"'] & A
\end{tikzcd}$$

induces a T-algebra $1 \xrightarrow{\tilde{c}} P \xrightarrow{\tilde{f}} P \cdots$ and hence a unique morphism

$it(\tilde{c}, \tilde{f}) : N \to P$. Further

$$
\begin{tikzcd}
N \arrow[r, "it(\tilde{c}, \tilde{f})"] \arrow[d, "1_N"] & P \arrow[d, hook] \\
N \arrow[r, "it(c, f)"'] & A
\end{tikzcd}
$$

commutes, by initiality of $(N, [z, s])$. This shows $(1_N, ([z, s], [z, s]) : 1_N \to 1_N$ is the initial $\overline{T}$-algebra. $\square$

In [Jac93, CS91] models of idts are required to satisfy a parameterisation property. A consequent requirement of parameterisation on the induction principle must be imposed. This requirement can be captured using the fibrations with indeterminates in §6.3. Details will appear elsewhere.

Chapter 5

Comonads and Kleisli fibration

5.1 Introduction

Categories with an indeterminate or generic element, also called polynomial categories, play an important role in the categorical interpretation of simply typed λ-calculus, as in [LS86]. Among other applications, they are used to express functional completeness properties of cartesian closed categories, *cf. ibid.* We set about doing a similar analysis for certain polymorphic λ-calculi in §6. In this chapter we develop the basic technical background necessary for this analysis. Specifically, we need a definition of Cartesian objects with an indeterminate element in a 2-category, generalising the formulation for ordinary categories with finite products, or cartesian categories, in [LS86]. We thus seek to instantiate such formulation in the 2-categories $\mathcal{F}ib(\mathbb{B})$ and $\mathcal{F}ib$, the 'universes' of models for polymorphic cal-culi. As we will show below, Kleisli objects for comonads play an important role in the construction of cartesian objects with an indeterminate. Thus, we are led to consider fibred comonads and Kleisli fibrations for them. These will be used in §6 to study so-called contextual and *functional* completeness of $\lambda \rightarrow$- and $\lambda\omega$-fibrations, the categorical versions of the polymorphic calculi $\lambda \rightarrow$ and $\lambda\omega$, along other applications of polynomial fibrations.

The structure of the chapter is as follows: in §5.2 we recall the definition of cartesian category with an indeterminate element and its presentation as a Kleisli category for a suitable comonad, primarily as a motivation for the technical developments in the rest of the chapter. §5.3 gives the 2-categorical version of Lambek's presentation of cartesian categories with an indeterminate element as Kleisli categories [LS86, Part I, Proposition 7.1]. This involves the reformulation, within a 2-category, of the usual operations on a cartesian category, §5.3.2. In §5.3.3 and §5.3.4 we carry out the reformulation in this general setting of the abovementioned result about objects with an indeterminate and Kleisli objects, Proposition 5.3.12. The more involved result is Lemma 5.3.11, which shows that the Kleisli object for a comonad on a cartesian object is again cartesian; although trivial in $\mathcal{C}at$, the 2-categorical version requires a special property 'PCK' of $\mathcal{C}at$, introduced in Definition 5.3.9, which holds in the 2-categories of fibrations as well.

In the remaining of the chapter, we deal with the existence of Kleisli objects for fibred comonads. In §5.4 we specialise the notion of comonad and resolution to the fibred case. In §5.4.1 we present Kleisli fibrations for comonads in $\mathcal{F}ib(\mathbb{B})$, which agree, both globally and fibrewise, with those in $\mathcal{C}at$. In §5.4.2 we consider resolutions for comonads in $\mathcal{F}ib$, which are not as simple as those for $\mathcal{F}ib$: we present a construction which 'factors' a fibred comonad through a resolution for its base comonad, based on the factorisation of fibred adjunctions of Theorem 3.2.3; this construction combined with the above construction in $\mathcal{F}ib(\mathbb{B})$ yields the required Kleisli fibrations. We make heavy use of the algebraic laws of fibred 2-cells, presented in §3.1.1.

The basic material on comonads in $\mathcal{C}at$ is taken from [Mac71, §VI] and [LS86, Part 0].

5.2 Categories with an indeterminate element

In this section we recall what it means for a category to have a generic global element or indeterminate element. More precisely, given a category $\mathbb{B}$ (with a terminal object 1) and an object $I \in \mathbb{B}$, we describe the so-called *polynomial category* $\mathbb{B}[x : I]$ obtained by freely adjoining a morphism $x : 1 \to I$ to $\mathbb{B}$ and characterise it in terms of a universal property. We follow [LS86, Part I, §5]. In §5.3.4 we will reformulate this universal property in an elementary 2-categorical way in order to apply the same formulation to fibrations in §6.2 and §6.3.

Assume $\mathbb{B}$ is a category with finite products and let $I \in |\mathbb{B}|$. We want a category with finite products $\mathbb{B}[x : I]$, with the objects and morphisms of $\mathbb{B}$ and an additional morphism $x : 1 \to I$, 1 the terminal object. We can construct $\mathbb{B}[x : I]$ as follows:

(i) Add an edge $x : 1 \to I$ to the underlying graph $\mathcal{G}(\mathbb{B})$ of $\mathbb{B}$, obtaining a graph $\mathcal{G}(\mathbb{B})[x : I]$.

(ii) Form the free category with finite products $\mathcal{F}(\mathcal{G}(\mathbb{B})[x : I])$ on this graph.

(iii) Make a suitable quotient, identifying morphisms, of $\mathcal{F}(\mathcal{G}(\mathbb{B})[x : I])$ so that the inclusion $\mathcal{G}(\mathbb{B}) \hookrightarrow \mathcal{G}(\mathbb{B})[x : I]$ becomes a finite product preserving functor $\eta : \mathbb{B} \to \mathbb{B}[x : I]$.

We think of the new morphism $x : 1 \to I$ above as a 'parameter of type I'. This means that x can be 'instantiated' by actual global elements. This is expressed categorically by the following universal property of $\mathbb{B}[x : I]$: for each category with finite products $\mathbb{C}$ and each finite product preserving functor $F : \mathbb{B} \to \mathbb{C}$ together with a morphism $F1 \xrightarrow{a} FI$ there is a unique finite product preserving functor $\overline{(F,a)} : \mathbb{B}[x : I] \to \mathbb{C}$ with $\overline{(F,a)}\eta = F$ and $\overline{(F,a)}x = a$ in

$$\begin{tikzcd}
\mathbb{B}[x : I] \arrow[r, dashed, "\overline{(F,a)}"] & \mathbb{C} \\
\mathbb{B} \arrow[u, "\eta"] \arrow[ur, "F"'] &
\end{tikzcd}$$

The functor $\overline{(F,a)}$ can be understood as performing 'substitution' of x by a. We will sometimes write η_I for η above, to make explicit the dependence on I. In [LS86, Part I,§6] this polynomial category is used to express a so-called *functional completeness* property of cartesian and cartesian closed categories, or simply typed λ-calculus. We will refine this notion and the corresponding generalisation to polymorphic λ-calculus in §6.1.

For a category $\mathbb{B}$ with finite products, an object I induces a comonad $\_ \times I : \mathbb{B} \to \mathbb{B}$, with counit at J given by $\pi_{I,J} : J \times I \to J$ and comultiplication given by $\langle 1, \pi'_{J,I} \rangle : J \times I \to (J \times I) \times I$. As shown in [LS86, Part I, Proposition 7.1], $B[x : I]$ is the Kleisli category $\mathbb{B}_{\_\times I}$ induced by this comonad. Furthermore, the functor $\eta : \mathbb{B} \to \mathbb{B}[x : I]$ corresponds to $U_{\_\times I} : \mathbb{B} \to \mathbb{B}_{\_\times I}$, the right adjoint of the Kleisli resolution of $\_ \times I$, and hence has a left adjoint. Also, when $\mathbb{B}$ is cartesian closed, we find in *ibid.* that $\mathbb{B}[x : I] \cong \mathbb{B}_{I \Rightarrow \_}$, for the monad $I \Rightarrow \_ : \mathbb{B} \to \mathbb{B}$ with unit at J given by $A(\pi_{J,I}) : J \to I \Rightarrow J$ (the adjoint transpose of the projection $\pi_{J,I}$) and multiplication $\Lambda(\mathrm{ev}_{I,J} \circ (\mathrm{ev}_{I,I \Rightarrow J} \times I) \circ \langle 1, \pi'_{I \Rightarrow (I \Rightarrow J), I} \rangle) : I \Rightarrow (I \Rightarrow J) \to I \Rightarrow J$. In this case, $\eta : \mathbb{B} \to \mathbb{B}[x : I]$ corresponds to $F_{I \Rightarrow \_} : \mathbb{B} \to \mathbb{B}_{I \Rightarrow \_}$, the left adjoint of the Kleisli resolution of $I \Rightarrow \_$, and hence has a right adjoint. The existence of a left (respectively right) adjoint to η corresponds to so-called *contextual* (respectively *functional*) completeness of $\mathbb{B}$ as explained in §6.1. We have the following result underlying the ones above; it is a variation of [BW85, §3.7,Theorem 5].

5.2.1. Proposition. *Given a comonad $G : \mathbb{C} \to \mathbb{C}$, consider its associated Kleisli resolution $F_G \dashv U_G : \mathbb{C} \to \mathbb{C}_G$. The following are equivalent:*

(i) G has a right adjoint $G \dashv T : \mathbb{C} \to \mathbb{C}$

(ii) U_G has a right adjoint $U_G \dashv R : \mathbb{C}_G \to \mathbb{C}$

Under either of the above equivalent hypotheses, $T (= RU_G)$ is the functor part of a monad and the corresponding Kleisli category $\mathbb{C}_T$ is isomorphic to

$\mathbb{C}_G$.

Proof. The equivalence is easily established, in view of the fact that T is part of a monad and hence induces a right adjoint R via its Kleisli resolution. The monad structure on T is induced as follows: let ϵ and δ be the counit and comultiplication respectively of the comonad G and let η' and ϵ' be the unit and counit of $G \dashv T$. The unit of the monad is $T\epsilon \circ \eta'$ and the multiplication is $T(\epsilon' \circ G\epsilon' T \circ \delta T^2) \circ \eta' T^2$. The isomorphism between the Kleisli categories $\mathbb{C}_G$ and $\mathbb{C}_T$ follows readily from

$$\mathbb{C}_G(X,Y) \cong \mathbb{C}(GX,Y) \cong \mathbb{C}(X,TY) \cong \mathbb{C}_T(X,Y)$$

□

5.2.2. Remark. The above proposition applies to any 2-category $\mathcal{K}$ which admits the construction of Kleisli objects for monads and comonads. The adjunction $G \dashv T$ establishes a one-to-one correspondence between oplax cocones for G and lax cocones for T.

5.3 Comonads and Kleisli objects in a 2-category

In this section we recall the concepts of comonad and its associated Kleisli object in a 2-category, following [Str72]. The purpose is to reformulate Lambek's presentation of the polynomial category $\mathbb{B}[x : I]$ — when $\mathbb{B}$ has finite products — as the Kleisli category of the comonad $\_ \times I : \mathbb{B} \to \mathbb{B}$, *cf.* §5.2, in a 2-category with suitable structure.

5.3.1. Definition. Given a 2-category $\mathcal{K}$, a *comonad* in it is a triple

$$\langle g : A \to A, \epsilon, \delta \rangle$$

where $\epsilon : g \Rightarrow 1_A$ and $\delta : g \Rightarrow g \circ g$ are called the *counit* and the *comultiplication* respectively. The data must satisfy

$$g\epsilon \circ \delta = 1_g = \epsilon g \circ \delta \qquad \delta g \circ \delta = g\delta \circ \delta$$

When $\mathcal{K} = \mathcal{C}at$, we get the usual notion of comonad. An adjunction $f \dashv u : A \to B$ via η, ϵ in $\mathcal{K}$ generates a comonad $\langle fu : A \to A, \epsilon, f\eta u \rangle$. In this case, $f \dashv u$ is a resolution for the comonad so generated, according to the following definition.

5.3.2. Definition. Given a comonad $\langle g : A \to A, \epsilon, \delta \rangle$ in $\mathcal{K}$, a *resolution* for it is an adjunction $f \dashv u$ via η', ϵ' such that

$$g = fu \qquad \epsilon' = \epsilon \qquad \delta = f\eta' u$$

In [LS86, Part 0], resolutions for a comonad in $\mathcal{C}at$ are organised into a category; a morphism between $f \dashv u : A \to B$ and $f' \dashv u' : A \to B'$ is a morphism $h : B \to B'$ such that $(h, 1_A)$ is a map of adjunctions, as in Definition 1.1.5. Every comonad in $\mathcal{C}at$ has an initial resolution with respect to that category. This resolution is given by the Kleisli category of the comonad, as in *ibid.* The corresponding notion of Kleisli object for a comonad in a 2-category is formulated in [Str72]. It amounts to an oplax colimit [Kel89]. We only give the definition of oplax colimit for a comonad, since this is the only instance we need.

5.3.3. Definition. Let $\langle g : A \to A, \epsilon, \delta \rangle$ be a comonad in $\mathcal{K}$.

(i) An *oplax cocone* (l, σ) consists of a morphism $l : A \to B$ and a 2-cell $\sigma : l \Rightarrow lg$, satisfying

$$l\epsilon \circ \sigma = 1_l \qquad \sigma g \circ \sigma = l\delta \circ \sigma$$

B is called the *vertex* of the cocone.

(ii) Given oplax cocones (l, σ) and (l', σ') with the same vertex, $l, l' : A \to B$, a morphism from (l, σ) to (l', σ') is a 2-cell $\gamma : l \Rightarrow l'$ satisfying $\gamma g \circ \sigma = \sigma' \circ \gamma$. We write $\gamma : (l, \sigma) \Rightarrow (l', \sigma')$ for such a morphism.

(iii) The above notion of morphism sets up a category $\mathcal{O}p\mathcal{L}ax(g, B)$ of oplax cocones with vertex B . A morphism $h : B \to C$ induces a functor $h \circ \_ : \mathcal{O}p\mathcal{L}ax(g, B) \to \mathcal{O}p\mathcal{L}ax(g, C)$. A 2-cell $\alpha : h \Rightarrow h'$ induces a natural transformation $\alpha \circ \_ : h \circ \_ \to h' \circ \_ : \mathcal{O}p\mathcal{L}ax(g, B) \to \mathcal{O}p\mathcal{L}ax(g, C)$.

(iv) An *oplax colimit* is an oplax cocone $(U : A \to A_g, \lambda)$ with the following universal property: there is an isomorphism

$$\mathcal{K}(A_g, B) \cong \mathcal{O}p\mathcal{L}ax(g, B)$$

2-natural in B.

The oplax colimit of a comonad is called its *Kleisli object.* When such objects exist for every comonad, we say that $\mathcal{K}$ *admits Kleisli objects for comonads.*

We refer to the object A_g itself as the Kleisli object.

5.3.4. Remarks.

- The above isomorphism means in elementary terms that, given an oplax cocone $(l : A \to B, \sigma)$, there is a unique morphism $\overline{(l, \sigma)} : A_g \to B$ such that $\overline{(l, \sigma)} \circ u = l$ and $\overline{(l, \sigma)}\lambda = \sigma$. The 2-dimensional aspect means that given a morphism $\gamma : (l, \sigma) \Rightarrow (l', \sigma')$, there is a unique 2-cell $\overline{\gamma} : \overline{(l, \sigma)} \Rightarrow \overline{(l', \sigma')}$ such that $\overline{\gamma}u = \gamma$.

- Any resolution $f \dashv u : A \to B$ for the comonad g induces an oplax cocone $(u, \eta u)$. As a partial converse, the oplax colimit (u, λ) is such that u has a left adjoint f and the adjunction $f \dashv u$ generates the comonad. See [Str72] for details.

Recall that in $\mathcal{C}at$ the Kleisli category $\mathbb{A}_G$ for a comonad G on $\mathbb{A}$ has the same objects as A and has hom-sets $\mathbb{A}_G(X, Y) = \mathbb{A}(GX, Y)$. Identities are given by instances of ϵ. For $f : GX \to Y$ and $g : GY \to Z$, their composite is $g \circ Gf \circ \delta_X$. There is an adjunction, written $F_G \dashv U_G : \mathbb{A} \to \mathbb{A}_G$ (via η, ϵ) which generates G. The induced oplax cocone $(U_G, \eta U_G)$ is an oplax colimit: given an oplax cocone $(L : \mathbb{A} \to \mathbb{B}, \sigma), \overline{(L, \sigma)} : \mathbb{A}_G \to \mathbb{B}$ is given by $\overline{(L, \sigma)}(f : GX \to Y) = Lf \circ \sigma_X : LX \to LY$, and a morphism $\gamma : (L, \sigma) \to (L', \sigma')$, induces $\overline{\gamma}_X = \gamma_X : \overline{(L, \sigma)}X \to \overline{(L', \sigma')}X$.

5.3.1 Products in a 2-category

In order to state that a category has binary products and terminal object in terms of adjunctions, we use the fact that the 2-category $\mathcal{C}at$ itself has finite products. Their definition in an arbitrary 2-category, from [Kel89], is as follows

5.3.5. Definition.

- A 2-category $\mathcal{K}$ has a *terminal object* if there is an object $\mathbf{1}$ such that for every object A, there is an isomorphism

  $$\mathcal{K}(A, \mathbf{1}) \cong \{*\}, \text{ the one-object one-morphism category}$$

  2-natural in A.

- $\mathcal{K}$ has *binary products* if for any two objects A and B, there is an object $A \times B$ such that, for any object C there is an isomorphism

  $$\mathcal{K}(C, A \times B) \cong \mathcal{K}(C, A) \times \mathcal{K}(C, B)$$

  2-natural in C.

We say that $\mathcal{K}$ has finite products if it has binary ones and a terminal object. The above isomorphisms mean that the underlying category $\mathcal{K}_0$, has finite products as an ordinary category, and that they have a 2-dimensional universal property. Specifically, $\mathbf{1}$ is such that for every object A, there is a unique 1-cell $!_A : A \to \mathbf{1}$ and a unique 2-cell $\alpha : !_A \Rightarrow !_A$, whence $\alpha = 1_{!_A}$.

Similarly, for objects A and B, the projections $A \xleftarrow{\pi} A \times B \xrightarrow{\pi'} B$ are such that for any span $A \xleftarrow{f} C \xrightarrow{g} B$ there is a unique $\langle f, g\rangle : C \to A \times B$ with $\pi \circ \langle f, g\rangle = f$ and $\pi' \circ \langle f, g\rangle = g$. And for any two 2-cells $\alpha : f \Rightarrow f' : C \to A$ and $\beta : g \Rightarrow g' : C \to B$ there is a unique 2-cell $\langle \alpha, \beta\rangle : \langle f, g\rangle \Rightarrow \langle f', g'\rangle$ with $\pi\langle \alpha, \beta\rangle = \alpha$ and $\pi'\langle \alpha, \beta\rangle = \beta$.

The non-elementary definition of products in $\mathcal{K}$ is given in terms of 2-adjoints: $\mathcal{K}$ has a terminal object if $!_{\mathcal{K}} : \mathcal{K} \to \{*\}$ has a right 2-adjoint; it has binary products if the diagonal 2-functor $\Delta : \mathcal{K} \to \mathcal{K} \times \mathcal{K}$ has a right 2-adjoint.

5.3.2 Cartesian objects in a 2-category with products

Rephrasing the definition of a category with finite products in $\mathcal{C}at$ in terms of adjoints, we get the following definition of cartesian objects in a 2-category with finite products [CKW90].

5.3.6. Definition. Let $\mathcal{K}$ be a 2-category with finite products. An object A is *cartesian* if both

- the unique morphism $!_A : A \to \mathbf{1}$ has a right adjoint $1 : \mathbf{1} \to A$, and
- the diagonal morphism $\delta_A : A \to A \times A$ has a right adjoint $\otimes : A \times A \to A$.

Note that the counit of $!_A \dashv 1$ must be the identity. If $\tau : 1_A \Rightarrow 1 \circ !_A$ is the unit, the adjunction laws reduce to $\tau 1 = 1_1$.

A cartesian object in $\mathcal{C}at$ is a category with assigned finite products. A cartesian object in $\mathcal{F}ib(\mathbb{B})$ a fibration with assigned fibred finite products, while a cartesian object in $\mathcal{F}ib$ is a fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ such that both $\mathbb{E}$ and $\mathbb{B}$ have assigned finite products and p preserves them strictly.

For the developments in §5.3.3 and §5.3.4, we need to spell out how the usual operations of pairing and projection in a category with finite products, as in [LS86, Part I], are obtained in this abstract setting.

In $\mathcal{C}at$, the projections associated to the binary product functor $\times : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ are natural transformations $\pi_{\_,\_} : \times \dot{\to} : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ and $\pi'_{\_,\_} : \times \dot{\to} \pi'$. Then, for objects X, Y of $\mathbb{A}$, $\pi_{X,Y} : X \times Y \to X$ is the first projection. The projections are the components of the counit $\epsilon' : \delta_{\mathbb{A}} \times \dot{\to} 1_{\mathbb{A} \times \mathbb{A}}$

Let A with $\delta_A \dashv \otimes : A \times A \to A$ via η', ϵ' be a cartesian object in $\mathcal{K}$. The associated projections are $p = \pi\epsilon' : \otimes \Rightarrow \pi$ and $q = \pi'\epsilon' : \otimes \Rightarrow \pi'$.

As for pairing, recall that objects of a category $\mathbb{A}$ correspond to functors $\mathbf{1} \to \mathbb{A}$, where $\mathbf{1}$ is the terminal category, and morphisms of $\mathbb{A}$ correspond to natural transformations between the respective functors. Given morphisms $f : Z \Rightarrow I : \mathbf{1} \to \mathbb{A}$ and $g : Z \Rightarrow J : \mathbf{1} \to \mathbb{A}$, their pairing $\langle f, g\rangle = (f \times g) \circ \delta_Z$. The diagonal morphism δ_Z is the component at Z of the unit $\eta' : 1_{\mathbb{A}} \dot{\to} \times\delta\mathbb{A}$.

Generalising to $\mathcal{K}$, given 'objects' $f, g : B \to A$ of A, their product is $\otimes\langle f, g\rangle : B \to A$. For 'morphisms' $\alpha : f \Rightarrow g : B \to A$ and $\beta : f \Rightarrow h : B \to A$, their pairing is $\langle\langle \alpha, \beta\rangle\rangle = \otimes\langle \alpha, \beta\rangle \circ \eta' f : f \Rightarrow \otimes\langle g, h\rangle$. Let $p_{g,h} = p\langle g, h\rangle : \otimes\langle g, h\rangle \Rightarrow g$ and $q_{g,h} = q\langle g, h\rangle : \otimes\langle g, h\rangle \Rightarrow h$. Then,

$$p_{g,h} \circ \langle\langle \alpha, \beta\rangle\rangle = \alpha \qquad q_{g,h} \circ \langle\langle \alpha, \beta\rangle\rangle = \beta \qquad \langle\langle p_{g,h}, q_{g,h}\rangle\rangle = 1_{\otimes\langle g,h\rangle}$$

Given cartesian objects A and B, with products $\otimes$ and $\hat{\otimes}$ respectively, a morphism $f : A \to B$ induces a 2-cell $\pi_f = \langle\langle fp, fq \rangle\rangle : f\otimes \Rightarrow \otimes'(f \times f)$ (the pairing is that of B). Then, f *preserves finite products* if ϕ is an isomorphism. This agrees in $\mathcal{C}at$ with the usual definition.

5.3.3 Comonad induced by a global element of a cartesian object

As we have seen, an object I of a category $\mathbb{A}$ with finite products induces a comonad $\_ \times I$ on it. This generalises 2-categorically as follows:

5.3.7. Definition. Let A be a cartesian object in $\mathcal{K}$, with $\delta_A \dashv \otimes$ via η', ϵ'. A global element $i : \mathbf{1} \to A$ induces a comonad $\langle g_i : A \to A, \epsilon, \delta \rangle$, where

- $g_i = \otimes\langle 1_A, i!_A \rangle$
- $\epsilon = p_{1_A, i!_A} (= \pi\epsilon'\langle 1_A, i!_A \rangle) : g_i \Rightarrow 1_A$
- $\delta = \langle\langle 1_{g_i}, q\langle 1_A, i!_A \rangle\rangle\rangle g_i 1_A (= \otimes\langle 1_\otimes, \pi'\epsilon' \rangle \langle 1_A, i!_A \langle \circ\ \eta' \otimes \langle 1_A, i!_A \rangle) : g_i \Rightarrow g_i^2$

The verification of the comonad laws proceeds by 2-categorical pasting; we omit the details.

5.3.4 Objects with an indeterminate

Given a category $\mathbb{B}$ with a terminal object 1, and any object I of $\mathbb{B}$, we recalled in §5.2 the universal property of $\mathbb{B}[x : I]$, the category with an indeterminate element of 'type' I. We also mentioned that, when $\mathbb{B}$ has finite products, $\mathbb{B}[x : I]$ could be presented as a Kleisli category. We give now the 2-categorical version of this result.

First, we must reformulate the 'category with an indeterminate' concept in a 2-category. Since we are interested in cartesian objects, we give a formulation of 'cartesian objects with an indeterminate'.

5.3.8. Definition. Let $\mathcal{K}$ be a 2-category with finite products. Let $\mathbb{B}$ be a cartesian object of $\mathcal{K}$ and let $i : \mathbf{1} \to B$ be a global element. The *cartesian object with an i-indeterminate* $B[x : i]$ is a cartesian object together with a morphism $j : B \to B[x : i]$ which preserves finite products and a 2-cell $x : j1 \Rightarrow ji : \mathbf{1} \to B[x : i]$ with the following universal property: given a cartesian object C, a finite product preserving morphism $f : B \to C$ and a 2-cell $\alpha : f1 \Rightarrow fi$, there is a unique finite product preserving morphism $\overline{(f, \alpha)} : B[x : i] \to C$ such that $\overline{(f, \alpha)}j = f$ and $\overline{(f, \alpha)}x = \alpha$. Further, given any other such pair (f', α') and a 2-cell $\gamma : f \Rightarrow f'$, there is a unique 2-cell $\overline{\gamma} : \overline{(f, \alpha)} \Rightarrow \overline{(f', \alpha)}$ such that $\overline{\gamma}j = \gamma$.

Now, we want to show that if $\mathcal{K}$ admits Kleisli objects for comonads, *i.e.* if the appropriate oplax colimits exist, the Kleisli object $B_{\_\otimes i}$ for the comonad $\_ \otimes i$ given in §5.3.3 has the universal property of $B[x : i]$.

We must show, among other facts, that $B_{\_\otimes i}$ is cartesian. In $\mathcal{C}at$, this follows from the fact that the Kleisli category $\mathbb{B}_G$, for a comonad G on $\mathbb{B}$ with finite products, has finite products. Consider objects X and Y in $\mathbb{B}_G$, then

$$\mathbb{B}_G(Z, X \times Y) \cong \mathbb{B}(GZ, X \times Y) \cong \mathbb{B}(GZ, X) \times \mathbb{B}(GZ, Y) \cong \mathbb{B}_G(Z, X) \times \mathbb{B}_G(Z, Y)$$

so products in $\mathbb{B}_G$ are obtained from those in $\mathbb{B}$. To generalise this to $\mathcal{K}$ assume the following property.

5.3.9. Definition. Let $\mathcal{K}$ be a category with finite products, which admits Kleisli objects for comonads. A comonad $\langle g : A \to A, \epsilon, \delta \rangle$ induces a comonad $\langle g \times g : A \times A \to A \times A, \epsilon \times \epsilon, \delta \times \delta \rangle$. Let $\langle u : A \to A_g, \lambda \rangle$ be the Kleisli object of g. $\mathcal{K}$ satisfies PCK if the oplax cocone $(U \times U : A \times A \to A_g \times A_g, \lambda \times \lambda)$ is an oplax colimit.

5.3.10. Remark. The above definition means that the Kleisli object of the product comonad $g \times g$ is given by the product of those for g. In $\mathcal{C}at$, we have

$$\begin{aligned}
(\mathbb{A} \times \mathbb{A})_{G \times G}((X, X'), (Y, Y')) &\cong \mathbb{A} \times \mathbb{A}((GX, GX'), (Y, Y')) \\
&\cong \mathbb{A}(GX, Y) \times \mathbb{A}(GX', Y') \\
&\cong \mathbb{A}_G(X, Y) \times \mathbb{A}_G(X', Y')
\end{aligned}$$

5.3.11. Lemma. *Let $\mathcal{K}$ satisfy* PCK. *Given a comonad $\langle g : A \to A, \epsilon, \delta \rangle$, if*

A is cartesian, so is A_g.

Proof. Let $(u : A \to A_g, \lambda)$ be the oplax colimit for g, and $!_A \vdash 1 : \mathbf{1} \to A$ via $\tau, 1$ and $\delta_A \vdash \otimes : A \times A \to A$ via η, ϵ be the adjunctions for the cartesian object A. The corresponding adjoints for A_g are written $!_{A_g} \vdash 1'$ via τ' and $\delta_{A_g} \vdash \otimes'$ via η', ϵ', as given below.

- Let $1' = u \circ 1$. $\eta' : 1_{A_g} \Rightarrow u1!_{A_g}$ is the unique such 2-cell induced by universality of (u, λ) as follows: 1_A is the unique mediating morphism from (u, λ) to itself, $u1!_{A_g}$ is the unique mediating morphism from (u, λ) to $(u1!_{A_g}u, u1!_{A_g}\lambda)$. Hence η' is determined by the oplax-cocone morphism $u\tau : u \Rightarrow u1!_{A_g}u$. The adjunction law $\eta' u1 = 1_{u1}$ follows by universality of the oplax cocone $(1_{\mathbf{1}}, 1_{1_{\mathbf{1}}})$ (for the identity comonad on $\mathbf{1}$), since $\eta' u1$ is uniquely determined by the oplax-cocone morphism $u\tau 1 = 1_{u1} : u1 \Rightarrow u1!_{A_g}u1$.

- To define $\otimes' : A_g \times A_g \to A_g$ we use the universal property of $(u \times u : A \times A \to A_g \times A_g, \lambda \times \lambda)$, guaranteed by PCK. Thus, we define an oplax cocone $(u \otimes : A \times A \to A_g, \sigma)$, where $\sigma = u\phi_g \circ \lambda \otimes$ as displayed below

$$\begin{tikzcd}
A \times A \arrow[r, "\otimes"] \arrow[d, "g \times g"'] \arrow[dr, phantom, "\Downarrow \phi_g"] & A \arrow[dr, "u"] \arrow[d, "g"] \arrow[r, phantom, "\Downarrow \lambda"] & \\
A \times A \arrow[r, "\otimes"'] & A \arrow[r, "u"'] & A_g
\end{tikzcd}$$

and $\phi_g = \langle\!\langle gp, gq \rangle\!\rangle$ is the 'comparison' 2-cell. This cocone induces the required $\otimes'$. To obtain the unit and counit of adjunction, we use the 2-dimensional property of the oplax colimits.

- The counit $\epsilon' : \delta_{A_g} \otimes' \Rightarrow 1_{A_g}$ is uniquely determined by the oplax-cocone morphism $(u \times u)\epsilon : (u \times u)\delta \otimes \Rightarrow u \times u$, from $((u \times u)\delta \otimes =)\delta u \otimes : A \times A \to A_g \times A_g, \delta\sigma)$ to $(u \times u, \lambda \times \lambda)$.
- The unit $\eta' : 1_{A_g} \Rightarrow \otimes' \delta_{A_g}$ is uniquely determined by the oplax-cocone morphism $u\eta : u \Rightarrow u \otimes \delta = \otimes' \delta u$, from (u, λ) to $(\otimes' \delta u : A \to A_g, \otimes' \delta\lambda)$
- $\otimes' \epsilon' \circ \eta' \otimes' = 1_{\otimes'}$ because: $\otimes' \epsilon'$ is uniquely determined by $u \otimes \epsilon : u \otimes \delta \otimes \Rightarrow u \otimes$ and $\eta' \otimes'$ is uniquely determined by $u\eta' \otimes : u \otimes \Rightarrow u \otimes \delta \otimes$. Thus, their composite is uniquely determined by the composite of these oplax-cocone morphisms, which is the identity by the adjunction laws for $\delta_A \dashv \otimes$.
- $\epsilon' \delta_{A_g} \circ \delta_{A_g} \eta' = 1_{\delta_{A_g}}$ because: $\epsilon' \delta_{A_g}$ is uniquely determined by $(u \times u)\epsilon\delta_A : (\delta_{A_g} u \otimes \delta_A, \sigma\delta) \Rightarrow (u\delta, \lambda\delta)$ and $\delta_{A_g}\eta'$ is uniquely determined by $(\delta u\eta =)(u \times u)\delta\eta : (\delta_{A_g} u, \delta_{A_g}\lambda) \Rightarrow (\delta_{A_g} \otimes' \delta_{A_g} u, \delta_{A_g} \otimes' \delta_{A_g}\lambda)$. Thus, their composite is uniquely determined by the composite of these oplax-cocone morphisms, which is the identity by the adjunction laws for $\delta_A \dashv \otimes$.

$\square$

Note that the construction of $1'$ and $\otimes'$ in the above proof is such that u

strictly preserves the assigned finite products 1 and $\otimes$.

5.3.12. PROPOSITION. *Let $\mathcal{K}$ be a 2-category with finite products, satisfying* PCK. *Let B be a cartesian object, with a global element $i : \mathbf{1} \to B$. B_{g_i} has the universal property of $B[x : I]$*

Proof. By Proposition 5.3.11, we have a finite product preserving morphism $u : B \to B_{g_i}$, with a 2-cell $x : u1 \Rightarrow ui$ given by the composite

$$\begin{tikzcd}
 & B \arrow[dd, "g_i"] \arrow[dr, "u"] & & \\
\mathbf{1} \arrow[ur, "1"] \arrow[dr, "i"'] \arrow[r, phantom, "\Downarrow q_{1,i}"] & {} \arrow[r, phantom, "\Downarrow \lambda"] & B_{g_i} & \\
 & B \arrow[ur, "u"'] & &
\end{tikzcd}$$

Given another cartesian object C, with product $\hat{\otimes}$, a finite product preserving morphism $f : B \to C$ and a 2-cell $\alpha : f1 \Rightarrow fi$, we have an oplax cocone on g_i (f, σ), where σ is the composite 2-cell in the diagram below

$$\begin{tikzcd}
 & C & \\
A \arrow[ur, "f"] \arrow[dr, "{\langle 1, i! \rangle}"'] \arrow[r, phantom, "\overset{\sigma'}{\Rightarrow}"] & C \times C \arrow[u, "\hat{\otimes}"] \arrow[r, phantom, "\overset{\phi^{-1}}{\Rightarrow}"] & A \arrow[ul, "f"'] \\
 & A \times A \arrow[u, "f \times f"] \arrow[ur, "\otimes"'] &
\end{tikzcd}$$

where $(\phi^{-1} =)\phi_f^{-1}$, the inverse of ϕ_f, exists because f preserves finite products, and $\sigma' = \langle\!\langle 1_f, \alpha!_A \circ f\tau \rangle\!\rangle$. Hence, by universality there exists a unique $\overline{(f,\sigma)} : B_{g_i} \to C$, such that $\overline{(f,\sigma)}\lambda = \sigma$. This implies $\overline{(f,\sigma)}x = \alpha$ as follows:

$$\begin{aligned}
\overline{(f,\sigma)}x &= fq_{1,i} \circ \sigma 1 \\
&= fq_{1,i} \circ \phi_f^{-1}\langle 1, i\rangle \circ \langle\!\langle 1_{f1}, \alpha \rangle\!\rangle \\
&= q_{f1,fi} \circ \langle\!\langle 1_{f1}, \alpha \rangle\!\rangle, \text{ because } \phi_f^{-1} \text{ is coherent w.r.t. projections} \\
&= \alpha
\end{aligned}$$

Finally, we must show $\overline{(f,\sigma)}$ preserves finite products. This holds because f preserves them: the comparison 2-cell : $\phi_{\overline{(f,\sigma)}} : \overline{(f,\sigma)} \otimes' \Rightarrow \hat{\otimes}(\overline{(f,\sigma)} \times \overline{(f,\sigma)})$ is uniquely determined by the morphism $\phi_f : (f\ \otimes, f\phi_g \circ \sigma\ \otimes) \Rightarrow (\hat{\otimes}(f \times f), \hat{\otimes}(\sigma \times \sigma))$. Then, $\phi_{\overline{(f,\sigma)}}$ is an isomorphism, since ϕ_f is. $\square$

5.4 Fibred comonads and resolutions

Instantiating Definition 5.3.1 in the 2-categories $\mathcal{F}\!ib$ and $\mathcal{F}\!ib(\mathbb{B})$, we get the appropriate notions of fibred comonad and $\mathbb{B}$-fibred comonad, respectively. We spell out the details only for $\mathcal{F}\!ib$; those for $\mathcal{F}\!ib(\mathbb{B})$ are obtained by considering the ap-propriate vertical instances.

5.4.1. Definition. A *fibred comonad* is given by the following data:

- a fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$

- a fibred 1-cell

$$\begin{tikzcd}
\mathbf{E} \arrow[r, "\tilde{G}"] \arrow[d, "p"'] & \mathbf{E} \arrow[d, "p"] \\
\mathbf{B} \arrow[r, "G"'] & \mathbf{B}
\end{tikzcd}$$

- fibred 2-cells $(\tilde{\epsilon}, \epsilon)$ and $(\tilde{\delta}, \delta)$

$$\begin{tikzcd}
\mathbf{E} \arrow[r, bend left, "1_E"] \arrow[r, bend right, "\tilde{G}"'] \arrow[r, phantom, "\Uparrow\tilde{\epsilon}"] \arrow[d, "p"'] & \mathbf{E} \arrow[d, "p"] \\
\mathbf{B} \arrow[r, bend left, "1_B"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Uparrow\epsilon"] & \mathbf{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{E} \arrow[r, bend left, "\tilde{G}\circ\tilde{G}"] \arrow[r, bend right, "\tilde{G}"'] \arrow[r, phantom, "\Uparrow\tilde{\delta}"] \arrow[d, "p"'] & \mathbf{E} \arrow[d, "p"] \\
\mathbf{B} \arrow[r, bend left, "G\circ G"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Uparrow\delta"] & \mathbf{B}
\end{tikzcd}$$

satisfying

$$\begin{array}{rclcrcl}
\tilde{G}\tilde{\epsilon} \circ \tilde{\delta} & = & 1_{\tilde{G}} & \qquad & G\epsilon \circ \delta & = & 1_G \\
\tilde{\epsilon}\tilde{G} \circ \tilde{\delta} & = & 1_{\tilde{G}} & & \epsilon G \circ \delta & = & 1_G \\
\tilde{G}\tilde{\delta} \circ \tilde{\delta} & = & \tilde{\delta}\tilde{G} \circ \delta & & G\delta \circ \delta & = & \delta G \circ \delta
\end{array}$$

As we can see from the above definition, a fibred comonad consists of a pair of comonads: the total one, $\langle \tilde{G} : \mathbb{E} \to \mathbb{E}, \tilde{\epsilon}, \tilde{\delta} \rangle$, and the base one $\langle G : \mathbb{B} \to \mathbb{B}, \epsilon, \delta \rangle$ such that the fibration $\begin{smallmatrix} \mathbb{E} \\ \downarrow p \\ \mathbb{B} \end{smallmatrix}$ is a morphism of monads, *i.e.* commutes with the counits and comultiplications, and $\tilde{G}$ is fibred over G. We will write $\langle (\tilde{G}, G) : p \to p, (\tilde{\epsilon}, \epsilon), (\tilde{\delta}, \delta) \rangle$ for such a fibred comonad, or briefly $(\tilde{G}, G)$.

A comonad in $\mathcal{F}ib(\mathbb{B})$ is a vertical fibred comonad, *i.e.* one where the base comonad is the identity such. It is therefore a $\mathbb{B}$-fibred comonad, but we drop the prefix when there is no ambiguity.

Similarly, instantiating Definition 5.3.1 in $\mathcal{F}ib$ we get:

5.4.2. Definition. A *fibred resolution* for $\langle (\tilde{G}, G) : p \to p, (\tilde{\epsilon}, \epsilon), (\tilde{\delta}, \delta) \rangle$ is a fibred adjunction

$$\begin{tikzcd}
\mathbb{E} \arrow[r, shift left=2, "\tilde{F}"] \arrow[r, phantom, "\perp"] \arrow[d, "p"'] & \mathbb{D} \arrow[l, shift left=2, "\tilde{U}"] \arrow[d, "q"] \\
\mathbb{B} \arrow[r, shift left=2, "F"] \arrow[r, phantom, "\perp"] & \mathbb{A} \arrow[l, shift left=2, "U"]
\end{tikzcd}$$

such that

$$\begin{array}{rclcrcl}
\tilde{F}\tilde{U} & = & \tilde{G} & \qquad & FU & = & G \\
\tilde{\epsilon}^{\tilde{F} \dashv \tilde{U}} & = & \tilde{\epsilon} & & \epsilon^{F \dashv U} & = & \epsilon \\
\tilde{F}\tilde{\eta}^{\tilde{F} \dashv \tilde{U}}\tilde{U} & = & \tilde{\delta} & & F\eta^{F \dashv U}U & = & \delta
\end{array}$$

where we have superscripted the units and counits with the corresponding adjunctions.

Thus, a fibred resolution for $(\tilde{G}, G)$ is a pair of resolutions for the corresponding comonads $\tilde{G}$ and G, with the resolution for $\tilde{G}$ fibred over the one for G, such that (p, q) is a map of adjunctions between these revolutions. Considering the appropriate vertical instance of the above definition, we get the notion of $\mathbb{B}$-fibred resolution.

The corresponding notions of fibred oplax cocone and fibred oplax colimit for the 2-categories $\mathcal{F}ib(\mathbb{B})$ and $\mathcal{F}ib$ are obtained similarly to those of comonad and resolution above. In the following section, we will show the existence of Kleisli fibrations for comonads in $\mathcal{F}ib(\mathbb{B})$ which are in turn used to build the corresponding ones in $\mathcal{F}ib$ in §5.4.2.

5.4.1 Kleisli fibration for a vertical fibred comonad

In this section we show how, generalising the situation in $\mathcal{C}at$, every comonad in $\mathcal{F}ib(\mathbb{B})$ has a Kleisli object, which we call its *Kleisli fibration.* If we consider fibrations as indexed categories or pseudo-functors, the usual construction in $\mathcal{C}at$ can be transferred to the present situation fibrewise. However it is simpler to present the construction globally, as we will show next.

5.4.3. PROPOSITION. *Given a fibred comonad $\langle G : p \to p, \epsilon, \delta \rangle$ for $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$, let $p_G : \mathbb{E}_G \to \mathbb{B}$ be the following functor:*

$$\begin{array}{rcl} p_G X & = & pX \\ p_G(f : GX \to Y & = & pf \end{array}$$

p_G is the Kleisli fibration for G via the oplax colimit induced by the (standard) resolution associated with the Kleisk category $\mathbb{E}_G$.

Proof.

- We first show p_G is a fibration. Given $u : I \to pX$ in $\mathbb{B}$, let $\overline{(u)}^{p_G}(X) = \overline{(u)}^{p}(X) \circ \epsilon_{u^{*p}(X)} : G(u^{*p}(X)) \to X$. Given a morphism $f : GY \to X$ with $pf = u \circ v$, for some $v : pY \to I$, there is a unique $\phi_f : GY \to u^{*p}(X)$ over v such that $\overline{(u)}^{p}(X) \circ \phi_f = f$. But then, the composite of ϕ_f and $\overline{(u)}^{p_G}(X)$ is

$$\overline{(u)}^{p}(X) \circ \epsilon_{u*p(X)} \circ G\phi_f \circ \delta_Y = \overline{(u)}^{p}(X) \circ \phi_f \circ G\epsilon_{u*p(X)} \circ \delta_Y = f$$

  by naturality of ϵ and the comonad laws. Therefore $\overline{(u)}^{p_G}(X)$ is cartesian in $\mathbb{E}_G$ and p_G is a fibration.

- The resolution $F_G \vdash U_G : \mathbb{E} \to \mathbb{E}_G$ is given by

$$\begin{array}{rclrcl} U_G X & = & X & U_G(X \xrightarrow{f} Y) & = & f \circ \epsilon_X \\ F_G X & = & GX & F_G(GX \xrightarrow{f} Y) & = & Gf \circ \delta_X \end{array}$$

  we then have a $\mathbb{B}$-fibred resolution $F_G \dashv U_G : p \to p_G$ since F_G and U_G preserve cartesian morphisms. The induced fibred oplax cocone $(U_G, \eta U_G)$ is a fibred oplax colimit: given a fibred oplax cocone $(L : p \to q, \sigma : L \Rightarrow LG)$, with $\begin{smallmatrix}\mathbb{D}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$, the unique mediating functor $\overline{(L, \sigma)} : \mathbb{E}_G \to \mathbb{D}$ in $\mathcal{C}at$ preserves cartesian morphisms:

$$\overline{(L,\sigma)}(\overline{(u)}^{p}(X) \circ \epsilon_{u*p(X)}) = L(\overline{(u)}^{p}(X)) \circ L(\epsilon_{u*p(X)}) \circ \sigma_{u*p(X)} = L(\overline{(u)}^{p}(X))$$

by the oplax cocone laws.

□

5.4.4. Remarks.

- The construction of Kleisli fibrations can also be presented fibrewise: for a fibred comonad G, the fibre $(\mathbb{E}_G)_I$ is $(\mathbb{E}_I)_{G|_I}$ for $G\mid_I\colon \mathbb{E}_I \to \mathbb{E}_I$ the ordinary comonad obtained by restriction. See Example 5.4.6. This fact is useful when dealing with (vertical) fibred structure, since then results for $\mathcal{C}at$ can be transferred to the fibred case in an straightforward fashion, *e.g.* Proposition 6.2.5.
- Similarly to the proof of the above proposition, we could show the existence of Kleisli fibrations for fibred monads, and Eilenberg-Moore objects, or objects of algebras [Str72], for fibred (co)monads. Again they agree globally and fibrewise with those in $\mathcal{C}at$.

Since Kleisli objects in $\mathcal{F}ib(\mathbb{B})$ are constructed as in $\mathcal{C}at$, we have as immediate consequence

5.4.5. Corollary. $\mathcal{F}ib(\mathbb{B})$ *satisfies* pck.

5.4.6. Example. Let $\langle G : \mathbb{C} \to \mathbb{C}, \epsilon, \delta\rangle$ be a comonad. It induces a $\mathcal{S}et$-fibred comonad $\langle Fam(G) : f(C) \to f(C), Fam(\epsilon), Fam(\delta)\rangle$ for the family fibration $f(C) : Fam(\mathbb{C}) \to \mathcal{S}et$ in an obvious fashion: the comonad on $\mathbb{C}^I$

(I a set) has action G^I . The Kleisli fibration for this 'family' comonad is simply $Fam(\mathbb{C}_T)$ This means that the 2-functor $Fam : \mathcal{C}at \to \mathcal{F}ib(\mathcal{S}et)$ preserves Kleisli objects for comonads. Similar considerations apply to monads and Kleisli and Eilenberg-Moore objects for them.

5.4.2 Kleisli fibration for a comonad in $\mathcal{F}ib$

In this subsection we construct Kleisli objects for comonads in $\mathcal{F}ib$. We call these objects *Kleisli fibrations.* The construction is based on the factorisation of fibred adjunctions given by Theorem 3.2.3; Kleisli objects in $\mathcal{F}ib$ are built from those in $\mathcal{C}at$ and $\mathcal{F}ib(\_)$.

Consider a fibred comonad $\langle(\tilde{G}, G) : p \to p, (\tilde{\epsilon}, \epsilon), (\tilde{\delta}, \delta)\rangle$ for $p : \mathbb{E} \to \mathbb{B}$ in $\mathcal{F}ib$, and a resolution for it, given by the following fibred adjunction

$$\begin{tikzcd}
\mathbb{D} \arrow[r, shift left, "\tilde{F}"] \arrow[r, phantom, "\perp"] \arrow[d, "q"'] & \mathbb{E} \arrow[l, shift left, "\tilde{U}"] \arrow[d, "p"] \\
\mathbb{A} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{B} \arrow[l, shift left, "U"]
\end{tikzcd}$$

This fibred adjunction can be factored, by Theorem 3.2.3, yielding an $\mathbb{A}$-fibred adjunction $\hat{F} \dashv \overline{U} : F^*(p) \to q$, and thus an $\mathbb{A}$-fibred comonad $\hat{F}\overline{U} : F^*(p) \to F^*(p)$. As we will show in Proposition 5.4.9 below, this $\mathbb{A}$-fibred comonad is determined by $F \dashv U$ and the comonad $(\tilde{G}, G)$. Then we can obtain the Kleisli fibration of $(\tilde{G}, G)$ as the Kleisli object for the $\mathbb{B}_G$-fibred comonad determined by $(\tilde{G}, G)$ and the Kleisli resolution for G, $F_G \dashv U_G : \mathbb{B}_G \to \mathbb{B}$; see Theorem 5.4.11.

To simplify the presentation, we consider the isomorphisms ϑ's between pull-backs, introduced in §3.1.1 are identities. This causes no lost of generality, by the $\mathcal{C}at$-fibred 2-equivalence $\mathcal{F}ib \equiv \mathcal{I}\mathcal{C}at$ of Corollary 1.3.10, since $\mathcal{I}\mathcal{C}at$ is split. So we work with $\mathcal{F}ib$ as if we were working with $\mathcal{I}\mathcal{C}at$; the property of having Kleisli objects is preserved under 2-equivalence.

5.4.7. WARNING. $\mathcal{F}ib$ does not have Kleisli objects in the sense of Definition 5.3.3, but only in a weaker, 'bicategorical' sense. This means that the oplax cocone (u, σ) we will construct below satisfies the universal property of the oplax colimit 'up to isomorphism'. See [Kel89] for a precise definition of bicategorical limits. This means that $\mathcal{F}ib$ admits the construction of fibrations with indeterminates in a similar weaker sense. This is a valid notion and is acceptable for the applications in §6. If we restrict attention to the 2-category $\mathcal{F}ib_{sp}$ of split fibrations and splitting-preserving morphisms, we do have Kleisli objects in the sense of Definition 5.3.3. We will give the definitions of the relevant constructions in $\mathcal{F}ib$, which involve coherent isomorphisms δ, ϕ given in §3.1.1, but in sketching the proof of the respective universal properties we ignore these isomorphisms, as if working in $\mathcal{F}ib_{sp}$ to simplify calculations.

Given a comonad $(\tilde{G}, G) : p \to p$ in $\mathcal{F}ib$ for $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$, and a resolution $F \dashv U : \mathbb{B} \to \mathbb{A}$ for the base comonad $G : \mathbb{B} \to \mathbb{B}$, we define an $\mathbb{A}$-fibred comonad $G_{F\dashv U}$ on $F^*(p)$ with the following property: given any fibred resolution for $(\tilde{G}, G)$ such that is base resolution is $F \dashv U$, then the fibred comonad it induces on $F^*(p)$ is (isomorphic to) $G_{F \dashv U}$.

5.4.8. Definition. Given a fibration $p : \mathbb{E} \to \mathbb{B}$, a fibred comonad on it $(\tilde{G}, G) : p \to p$ (with counit $(\tilde{\epsilon}, \epsilon)$ and comultiplication $(\tilde{\delta}, \delta)$) and a resolution $F \dashv U : \mathbb{B} \to \mathbb{A}$ (via η, ϵ) for $G : \mathbb{B} \to \mathbb{B}$, its *associated* $\mathbb{A}$*-fibred comonad* $G_{F \dashv U} : F^*(p) \to F^*(p)$ with counit $\overline{\epsilon}$ and comultiplication $\overline{\delta}$ is given as follows:

- $G_{F \dashv U} = \langle F\eta \rangle_p F^*(\hat{G})$. Recall that $\hat{G} : p \to G^*(p)$ is obtained by factoring $\tilde{G} : \mathbb{E} \to \mathbb{E}$ through the pullback of G and p, and $\langle F\eta \rangle_p (UF)^*(p) \to p$ is the $\mathbb{A}$-fibred 1-cell induced by $F\eta$ as in Lemma 3.1.2.(i).
- $\overline{\epsilon} = \delta^p_{F\eta, \epsilon F} \circ \langle F\eta \rangle_p F^*(\hat{\epsilon})$, where $\hat{\epsilon} : \hat{G} \Rightarrow \langle \epsilon \rangle_p$, is the $\mathbb{B}$-fibred 2-cell obtained by factoring $\tilde{\epsilon} : \tilde{G} \Rightarrow 1_E$ through $\epsilon : G \Rightarrow 1_B$ as in Lemma 3.1.2.(ii).
- $\overline{\delta} = (\phi^{\overline{G}}_\eta)^{-1} F^*(\hat{G}) \circ \langle F\eta \rangle_p F^*(\hat{\delta})$, where $\hat{\delta} : \hat{G} \Rightarrow \langle \delta \rangle_p \circ G^*(\hat{G}) \circ \hat{G}$ is the $\mathbb{B}$-fibred 2-cell obtained by factoring $\tilde{\delta} : \tilde{G} \Rightarrow \tilde{G}^2$ through $\delta : G \Rightarrow G^2$ as in Lemma 3.1.2.(ii).

We must verify that the data in the definition above yields an $\mathbb{A}$-fibred comonad. The comonad laws for $\overline{G}$ follow from those of $(\tilde{G}, G)$, using the lemmas in §3.1.1 and Lemma 3.1.2. We show $\overline{G}\overline{\epsilon} \circ \overline{\delta} = 1_{\overline{G}}$ for illustration. First,

$$(\phi^{\overline{G}}_\eta)^{-1} = \langle F\eta \rangle_p (\phi^{\hat{G}}_{F\eta})^{-1} \circ ((\delta^p_{F\eta, GF\eta})^{-1} \circ \delta^p_{F\eta, \delta F}) F^*(G^*(\hat{G}))$$

by Lemma 3.1.2.(ii), with the same kind of 'pasting' argument as explained for step (#) below. Then, omitting some subscripts and superscripts for brevity,

$$
\begin{array}{rl}
 & \overline{G}\overline{\epsilon} \circ \overline{\delta} \\
= & \langle F\eta\rangle F^*(\hat{G})\delta_{F\eta,\epsilon F} \circ \langle F\eta\rangle F^*(\hat{G})\langle F\eta\rangle F^*(\hat{\epsilon}) \circ (\phi^{\overline{G}_\eta})^{-1}\langle F\eta\rangle F^*(\hat{\delta}) \\
= & \langle F\eta\rangle F^*(\hat{G})\delta_{F\eta,\epsilon F} \circ \langle F\eta\rangle (\phi^{\hat{G}}_{F\eta})^{-1}\langle \epsilon F\rangle F \circ \\
 & ((\delta^p_{F\eta,GF\eta})^{-1}\delta^p_{F\eta,\delta F})F^*(G^*(\hat{G}))F^*(\hat{G}) \circ \langle F\eta\rangle F^*(\langle\delta\rangle G^*(\hat{G})\hat{G}\epsilon \circ \hat{\delta}), \\
 & \text{using the above expression for } (\phi^{\overline{G}}_\eta)^{-1} \text{ and the interchange law for 2-cells} \\
= & \langle F\eta\rangle \delta_{F\eta UF,G\epsilon F}F^*(\hat{G}) \circ \langle\delta F\rangle \phi^{\hat{G}}_{\epsilon F} \circ \langle F\eta\rangle F^*(\langle\delta\rangle G^*(\hat{G})\hat{G}\epsilon \circ \hat{\delta}), \\
 & (\#); \text{see explanation below} \\
 & \langle F\eta\rangle F^*(\delta_{F\eta U,G\epsilon})F^*(\hat{G}) \circ (\langle\delta\rangle)F^*(\phi^{\hat{G}}_\epsilon) \circ \langle F\eta\rangle F^*(\langle\delta\rangle G^*(\hat{G})\hat{G}\hat{\epsilon} \circ \hat{\delta}), \\
 & \text{using Lemmas 3.1.6, 3.1.10.(iii) and 3.1.2.(ii)} \\
= & \langle F\eta\rangle F^*(\delta_{F\eta U,G\epsilon} \circ \langle\delta\rangle(\phi^{\hat{G}}_\epsilon \circ G^*(\hat{G})\hat{G}\hat{\epsilon}) \circ \hat{\delta}), \\
= & \langle F\eta\rangle F^*(\delta_{F\eta U,G\epsilon} \circ \langle\delta\rangle\widehat{(\tilde{G}\tilde{\epsilon})} \circ \hat{\delta}), \\
 & \text{by Lemma 3.1.9.(v)} \\
= & \langle F\eta\rangle F^*(((\widehat{\tilde{G}\tilde{\epsilon} \circ \tilde{\delta}}), \\
 & \text{by Lemma 3.1.9.(ii)} \\
 & 1_{\langle F\eta\rangle F^*(\hat{G})}
\end{array}
$$

Step (#) above uses

$$
\begin{array}{l}
\langle F\eta\rangle\delta_{F\eta UF,G\epsilon F}F^*(\hat{G}) \circ \langle F\eta\rangle\langle\delta F\rangle\phi^{F^*(\hat{G})}_{\epsilon F} = \\
\quad \langle F\eta\rangle(F^*(\hat{G})\delta_{F\eta,\epsilon F} \circ (\phi^{F^*(\hat{G})}_{F\eta})^{-1}\langle\epsilon F\rangle) \circ (\delta^{-1}_{F\eta,FUF\eta} \circ \delta_{F\eta,\delta F})F^*(G^*(\hat{G})\langle\epsilon F\rangle
\end{array}
$$

which is proved equating both sides by 'pasting' with $(F\eta)'_p\langle\delta F\rangle\langle\epsilon F\rangle \circ p^*(GF)(\epsilon F \circ \delta F)'_p$ and applying Lemma 3.1.2.(ii).

5.4.9. Proposition. *Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration, $(\tilde{G}, G) : p \to p$ be a comonad in $\mathcal{F}ib$, with counit $(\tilde{\epsilon}, \epsilon)$ and comultiplication $(\tilde{\delta}, \delta)$, and $(\tilde{F}, F) \dashv (\tilde{U}, U) : p \to q$, with $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{A}\end{smallmatrix}$, be a resolution for it. Let $\hat{F} \dashv \overline{U} : F^*(p) \to q$ be the $\mathbb{A}$-fibred adjunction induced by this resolution, as in Theorem 3.2.3. Then, the comonad induced by $\hat{F} \dashv \overline{U}$ on $F^*(p)$ is isomorphic to $G_{F \dashv U}$.*

Proof. We show the argument for $\mathcal{F}ib_{sp}$ to simplify the presentation. Recall from the proof of Theorem 3.2.3 that $\hat{F} \dashv \overline{U} : F^*(p) \to q$ is given by:

- $\overline{U} = \langle \eta \rangle_q \circ F^* \hat{U}$, where $\hat{U}$ is obtained by factoring $\tilde{U}$ thorugh the pullback of U and p.
- The unit $\overline{\eta} : 1 \Rightarrow \langle \eta \rangle_q F^* \hat{U} \hat{F}$ is the vertical factor of the unit $(\tilde{\eta}, \eta) : 1 \Rightarrow (\tilde{U}, U) \circ (\tilde{F}, F)$.
- The counit $\overline{\epsilon} : 1 \Rightarrow \hat{F} \langle \eta \rangle_q F^*(\hat{U})$ is the vertical factor of the fibred 2-cell $(\tilde{\epsilon} p^*(F) \circ \tilde{F} \eta' F^*(\hat{U}), \epsilon F \circ F \eta) : (\tilde{F} \langle \eta \rangle_q F^*(\hat{U}), F) \Rightarrow (p^*(F), F)$.

The comonad induced by $\hat{F} \dashv \overline{U} : F^*(p) \to q$ on $F^*(p)$ is then given by the following data:

- The comonad functor is

  $$\hat{F} \circ \overline{U} = \hat{F} \circ \langle \eta \rangle_q \circ F^* \hat{U} = \langle F\eta \rangle_p \circ F^*(U^*(\hat{F})) \circ F^*(\hat{U}) = \langle F\eta \rangle_p \circ F^*(\hat{T})$$

  where the second step results from Lemma 3.1.5 and the last step from a routine diagram chase.

- The counit is

$$\overline{\epsilon} = \langle F\eta\rangle_p F^*(\hat{\epsilon}) \circ 1_{\langle\eta\rangle_{F^*(p)} \circ F^*(\hat{T})} = \langle F\eta\rangle_p F^*(\hat{\epsilon})$$

  applying Lemma 3.1.9.(ii) to the definition of $\overline{\epsilon}$ given above and taking into account that $\tilde{F}\eta' F^*(\hat{U})$ is a cartesian 2-cell and hence its vertical factor is (isomorphic to) $1_{\langle\eta\rangle_{F^*(p)} \circ F^*(\hat{T})}$

- The comultiplication is

$$\begin{array}{rcll}
\overline{\delta} & = & \hat{F} \circ \hat{\eta_q} \circ F^*(\hat{U}) & \\
 & = & \widehat{\hat{F}\tilde{\eta}}_{F^*(p)} \circ \langle\eta\rangle_q \circ F^*(\hat{U}) & \text{by Lemma 3.1.8.(i)} \\
 & = & \widehat{\tilde{F}\tilde{\eta}_p} \circ \langle\eta\rangle_q \circ F^*(\hat{U}) & \text{by change-of-base} \\
 & = & \langle F\eta\rangle_p \circ (UF)^*\widehat{\tilde{F}\tilde{\eta}_p}) \circ F^*(\hat{U}) & \text{by Lemma 3.1.8.(ii)} \\
 & = & \langle F\eta\rangle_p \circ F^*(U^*(\widehat{\tilde{F}\tilde{\eta}_p}) \circ \hat{U}) & \\
 & = & \langle F\eta\rangle_p \circ F^*(\widehat{\tilde{F}\tilde{\eta}U}) & \text{by Lemma 3.1.9.(iv)} \\
 & = & \langle F\eta\rangle_p \circ F^*(\hat{\delta}) &
\end{array}$$

This argument provides an alternative proof that $\langle\overline{G}, \overline{\epsilon}, \overline{\delta}\rangle$ is a comonad. $\square$

We now show how to construct the Kleisli fibration for a comonad in $\mathcal{F}ib$, following the steps outlined at the beginning of this subsection. To structure the proof, we prove the following lemma about 'reindexing' of oplax cocones.

5.4.10. LEMMA. *Given*

- *a fibred comonad* $\langle(\tilde{G}, G) : p \to p, (\tilde{\epsilon}, \epsilon), (\tilde{\delta}, \delta)\rangle$ *for* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$,

- *a fibred oplax cocone* $((\tilde{L}, L) : p \to q, (\tilde{\sigma}, \sigma))$*, with* $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{C}\end{smallmatrix}$

- *an oplax cocone* $(K : \mathbb{B} \to \mathbb{A}, \nu : K \Rightarrow KG)$ *for* G*, and*

- *a functor* $J : \mathbb{A} \to \mathbb{C}$ *such that* $JK = L$ *and* $J\nu = \sigma$*. There is a unique oplax cocone* $(L' : \mathbb{E} \to J^*(\mathbb{D}), \sigma^\dagger : L' \Rightarrow L'\tilde{G})$ *such that* $((L', K), (\nu, \sigma))$ *is a fibred oplax cocone for* $(\tilde{G}, G), q^*(J)L' = L$ *and* $q^*(J)\sigma^\dagger = \tilde{\sigma}$*.*

Proof. Let $L' = \langle Kp, \tilde{L} \rangle : \mathbb{E} \to J^*(\mathbb{D})$ be the unique functor into the pullback $J^*(\mathbb{D})$. Then $q^*(J)L' = L$ holds. Also, $\langle KGp, \tilde{L}\tilde{G} \rangle = L'\tilde{G} : \mathbb{E} \to J^*(\mathbb{D})$ is the unique such functor. Hence the 2-cells $\tilde{\sigma} : \tilde{L} \Rightarrow \tilde{L}\tilde{G}$ and $\sigma p : Kp \Rightarrow KGp$ determine a 2-cell $\sigma^\dagger = \langle \tilde{\sigma}, \sigma p \rangle : L' \Rightarrow L'\tilde{G}$. Then, $q^*(J)\sigma^\dagger = \tilde{\sigma}$ holds. It only remains to verify that $(L', \sigma^\dagger)$ is an oplax cocone for G; the rest is immediate.

$L'\tilde{\epsilon} \circ \sigma^\dagger = 1_{L'}$, by the universal property of the pullback:

- $J^*(q)(L'\tilde{\epsilon} \circ \sigma^\dagger) = K\epsilon p \circ \sigma p = 1_{J^*(q)L'}$, because (K, σ) is an oplax cocone, and

- $q^{ast}(J)(L'\tilde{\epsilon} \circ \sigma^\dagger) = \tilde{L}\tilde{\epsilon} \circ \tilde{\sigma} = 1_{q^*(J)L'}$, because $(\tilde{L}, \tilde{\sigma})$ is an oplax cocone.

A similar argument shows $L'\tilde{\delta} \circ \sigma^\dagger = \sigma^\dagger\tilde{G} \circ \sigma^\dagger$.

5.4.11. Theorem. *Given a fibred comonad* $\langle (\tilde{G}, G) : p \to p, (\tilde{\epsilon}, \epsilon), (\tilde{\delta}, \delta) \rangle$ *for* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *, let* $\langle \overline{G} = G_{F_G \dashv U_G} : F_G^*(p) \to F_G^*(p), \overline{\epsilon}, \overline{\delta} \rangle$ *be the associated* $\mathbb{B}_G$*-fibred comonad induced by the Kleisli resolution* $F_G \dashv U_G : \mathbb{B} \to \mathbb{B}_G$*, via* η, ϵ*, for* G*. Then,* $F_G^*(p)_{\overline{G}} : F_G^*(\mathbb{E})_{\overline{G}} \to \mathbb{B}_G$ *is the fibration of the Kleisli object for* $(\tilde{G}, G)$ *in* $\mathcal{F}ib$*.*

Proof. In the proof, we omit $_G$ subscripts from F and U above and write p' for $F_G^*(p)$.

- By Lemma 3.2.1, we have a fibred adjunction

$$\begin{tikzcd} F^*(\mathbf{E}) \arrow[d, "p'"'] \arrow[r, shift left, "p^*(F)"] \arrow[r, phantom, "\perp"] & \mathbf{E} \arrow[l, shift left, "\overline{U}"] \arrow[d, "p"] \\ \mathbf{A} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{B} \arrow[l, shift left, "U"] \end{tikzcd}$$

  with $\overline{U} = (p')^*(U)\langle\epsilon F\rangle$ counit $(\epsilon')_p$ and unit $((\eta)'_{p'}\langle\epsilon F\rangle_q) \circ (\delta^p_{F\eta,\epsilon F})$. Let $(U' : p' \to p'_{\overline{G}}, \lambda : U' \Rightarrow U'\overline{G})$ be the Kleisli object of $\overline{G}$ in $\mathcal{F}ib(\mathbb{B}_G)$.' We then have an oplax cocone $((U'\overline{U}, U) : p \to p'_{\overline{G}}, (\overline{\lambda}, \eta U))$ for $(\tilde{G}, G)$ in $\mathcal{F}ib$, where $\overline{\lambda} = (U'((\eta)'_{p'}\langle\epsilon F\rangle_p\overline{G}) \circ \lambda)\overline{U}$, unfolding the definition of the 2-cells involved.

- To verify $((U'\overline{U}, U), (\overline{\lambda}, \eta U))$ is a fibred oplax colimit, let $((\hat{L}, L) : p \to q, (\tilde{\sigma}, \sigma))$, with $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{C}\end{smallmatrix}$ be another fibred oplax cocone.

  - Since $(U, \eta U)$ is an oplax colimit, there is a unique morphism $H = \overline{(L, \sigma)} : \mathbb{B}_G \to \mathbb{C}$ with $HU = L$ and $H\eta U = \sigma$. By Lemma 5.4.10, there is a fibred oplax cocone $((L', U) : p \to H^*(q), (\sigma^\dagger, \eta U))$. We write q' for $H^*(q)$.

  - Let $\hat{\sigma} = \hat{L}' \Rightarrow \langle\eta U\rangle_{q'} G^*(\hat{L}')\hat{L}'$ be the vertical factor of $\sigma^\dagger$ obtained by Lemma 312.(ii), so that $\sigma^\dagger = (\eta U)'_{q'}, G^*(\hat{L}')\hat{L}' \circ (q')^*(U)\hat{\sigma}$.

Then, $(\langle\eta\rangle_{q'}F^*(\hat{L}') : p' \to q', \langle\eta\rangle_{q'}F^*(\hat{\sigma}))$ is an oplax cocone. Note that the codomain of $\langle\eta\rangle_{q'}F^*(\hat{\sigma})$ is

$$\begin{array}{rl} & \langle\eta\rangle_{q'}F^*\langle\eta U\rangle_{q'}F^*(G^*(\hat{L}'))F^*(\hat{G}) \\ = & \langle\eta\rangle_{q'}\langle UF\eta\rangle_{q'} \circ (GF)^*(\hat{L}')F^*(\hat{G}), \\ & \text{by Lemma 3.1.10.(iii) and naturality of } \eta \\ = & \langle\eta\rangle_{q'}F^*(\hat{L}')\langle F\eta U\rangle_p F^*(\hat{G}), \\ & \text{by Proposition 3.1.5} \end{array}$$

and so the above oplax cocone is well-defined. The oplax cocone laws follow from those of $(L', \sigma^\dagger)$. We show one for illustration:

$$\begin{array}{rl} & \langle\eta\rangle_{q'}F^*(\hat{L}')\overline{\epsilon} \circ \langle\eta\rangle_{q'}F^*(\hat{\sigma}) \\ = & \langle\eta\rangle_{q'}(F^*(\hat{L}')\langle F\eta\rangle_p F^*(\hat{\epsilon}) \circ F^*(\hat{\sigma})) \\ = & \langle\eta\rangle_{q'}(F^*(\hat{L}')\langle F\eta\rangle_{U^*(q')}(GF)^*(\hat{L}')F^*(\hat{\epsilon}) \circ F^*(\hat{\sigma})) \\ & \text{by Proposition 3.1.5} \\ = & \langle\eta\rangle_{q'}(\langle\eta UF\rangle_{q'}(GF)^*(\hat{L}')F^*(\hat{\epsilon}) \circ F^*(\hat{\sigma})) \\ & \text{by Lemma 3.1.10.(iii) and naturality of } \eta \\ = & \langle\eta\rangle_{q'}F^*(\langle\eta U\rangle_{q'}G^*(\hat{L}')\hat{\epsilon} \circ \hat{\sigma}) \\ & \text{by Lemma 3.1.10.(iii) and the interchange law} \\ = & \langle\eta\rangle_{q'}F^*\widehat{((L'\tilde{\epsilon} \circ \sigma^\dagger)}) \\ & \text{by Lemma 3.1.9.(ii)} \\ & 1_{\langle\eta\rangle_{q'}F^*(\hat{L}')} \\ & \text{because}(L', \sigma^\dagger) \text{ is an oplax cocone} \end{array}$$

– By the universal property of $(U' : p' \to p'_{\overline{G}}, \lambda : U' \Rightarrow U'\overline{G})$ we have a unique oplax-cocone morphism $H' : p'_{\overline{G}} \to q'$ with $H'U' = \langle\eta\rangle_{q'}F^*(\hat{L}')$ and $H'\lambda = \langle\eta\rangle_{q'}F^*(\sigma))$ which is also a fibred oplax-cocone morphism from $((U'\overline{U}, U), (\overline{\lambda}, \eta U))$ to $(L', \sigma^\dagger)$. To see this, with $\sigma^\dagger = (\eta U)'_q G^*(\hat{L})\hat{L}' \circ (q')^*(U)\hat{\sigma}$ we must show $H'\overline{\lambda} = \sigma^\dagger$, which follows from

$$
\begin{array}{rl}
& H'U'(\eta)'_{p'}\langle\epsilon F\rangle_p\overline{GU} \\
= & \langle\eta\rangle_{q'}F^*(\hat{H})(\eta)'_p\langle\epsilon F\rangle_p\overline{GU} \\
= & (\eta U)'_q(UFU)^*(\langle\eta\rangle_{q'})\langle\epsilon\rangle G^*U^*(q')U^*(\langle F\eta\rangle_{U^*(q')})\langle\epsilon\rangle_{G^*U^*(q')}G^*(\hat{H})\hat{H} \\
& \text{by Proposition 3.1.5} \\
= & (\eta U)'_{q'}\langle UF\eta U\rangle_{q'}\langle UFU\epsilon\rangle_{q'}\langle UF\eta U\rangle_{q'}\langle UFU\epsilon\rangle_{q'}G^*(\hat{H})\hat{H} \\
& \text{using Lemma 3.1.10} \\
= & (\eta U)'_{q'}G^*(\hat{H})\hat{H}) \\
& \text{by the adjunction laws}
\end{array}
$$

and

$$
\begin{array}{rl}
& H'\lambda\overline{U} \\
= & \langle\eta\rangle_{q'}F^*(\sigma))\overline{U} \\
= & (q')^*(U)U^*(\langle\eta\rangle_{q'})U^*(F^*(\tilde{\sigma})\langle\epsilon\rangle_p \\
& \text{by diagram chase with pullbacks} U^*(\_) \\
= & (q')^*(U)U^*(\langle\eta\rangle_{q'})\langle\epsilon\rangle_{U^*(q')}\hat{\sigma} \\
& \text{using Lemma 3.1.8.(ii)} \\
= & (q')^*(U)\langle\eta U\rangle_{q'}\langle U\epsilon\rangle_{q'}\hat{\sigma}) \\
= & (q')^*(U)\hat{\sigma}) \\
& \text{by the adjunction laws}
\end{array}
$$

– We thus get a unique fibred 1-cell $(q^*(H)H', H) : p_{\overline{G}} \to q$ which makes $((U'\overline{U}, U), (\overline{\lambda}, \eta U))$ a fibred oplax colimit.

□

5.4.12. Remark. The above process to build the Kleisli fibration can be applied to obtain the Eilenberg-Moore one as well, starting with the Eilenberg-Moore resolution for the base comonad. In this case, this rather involved construction admits a simpler presentation: the Eilenberg-Moore fibration for a fibred comonad $\langle(\tilde{G}, G) : p \to p, (\tilde{\epsilon}, \epsilon), (\tilde{\delta}, \delta)\rangle$ for $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ is the fibration $p^{\tilde{G},G} : \mathbb{E}^{\tilde{G}} \to \mathbb{B}^G$, where $\mathbb{E}^{\tilde{G}}$ and $\mathbb{B}^G$ are the Eilenberg-Moore categories for $\tilde{G}$ and G respectively, and

$$p^{\tilde{G},G}(X \xrightarrow{x} \tilde{G}X) = (p(X) \xrightarrow{px} Gp(X))$$

The construction of Kleisli objects in $\mathcal{F}ib$ from those in $\mathcal{C}at$ and $\mathcal{F}ib(\_)$ yield at once the following corollary:

5.4.13. Corollary. *$\mathcal{F}ib$ satisfies* pck.

Chapter 6

Indeterminates in polymorphic λ-calculi

In this chapter we apply the constructions of Kleisli fibrations in $\mathcal{F}ib(\mathbb{B})$, Proposition 5.4.3, and $\mathcal{F}ib$, Theorem 5.4.11, to obtain fibrations with an indeterminate element, according to Proposition 5.3.12. We will show that this construction is adequate for $\lambda\rightarrow$- and $\lambda\omega$-fibrations. This will allow us to show contextual and functional completeness for these calculi, as explained below. Another application of these fibrations with indeterminates is to give a semantics for ML-style module features: signatures, structures and functors [HMM86], following the approach in [FP92]. The contents of this chapter is borrowed from [HJ93], although here we are concerned with a structural presentation on how the polynomial categories obtained as Kleisli fibrations inherit the relevant structure to interpret polymorphic calculi.

The structure of the chapter is as follows: in §6.1 we refine Lambek's analysis [LS86, Part I,§6] of representability of terms with a parameter, *i.e*, morphisms in a polynomial category, introducing *contextual completeness*, characterised by the existence of a certain left-adjoint and understand *functional completeness* in a dual fashion. This is extended from simply typed to polymorphic λ-calculi. §6.2 deals with indeterminates for fibrations over $\mathbb{B}$, establishing contextual and functional completeness for the several polymorphic calculi in $\mathcal{F}ib(\mathbb{B})$. It also presents the *simple* fibration of [Jac91a, §1.2.7] as a Kleisli fibration. §6.3 deals with indeterminates for fibrations in $\mathcal{F}ib$ and establishes functional completeness for $\lambda\omega$ in this context. Finally, §6.3.1 shows how polynomial fibrations can be used to give semantics to ML-style module-features: signatures, structures and functors.

6.1 Contextual and functional completness for λ-calculi

The categorical concept of 'category with an indeterminate element' or polynomial category, as in §5.2, captures the notion of parameterisation. It can be formulated type-theoretically as follows: given a simply typed λ-calculus $\mathcal{L}$, consider the simply typed λ-calculus $\mathcal{L}(c)$ obtained from $\mathcal{L}$ by freely adjoining a new constant $c : \sigma$; its typing relation $\vdash_c$ extends that of $\mathcal{L}$, $\vdash$, by $\vdash_c c : \sigma$. Thus, the terms of $\mathcal{L}(c)$ have a parameter $c : \sigma$.

We can then formulate certain 'representability' properties of $\mathcal{L}$. Specifically, we can consider contextual and functional completeness, which express the two ways a term $\Gamma \vdash_c t : \tau$ of $\mathcal{L}(c)$ can be represented by a term of $\mathcal{L}$:

(i) By a unique term $\Gamma, x : \sigma \vdash \llcorner t \lrcorner : \tau$ (same type, extended context) such that $\Gamma \vdash_c t = \llcorner t \lrcorner[x := c] : \tau$.

(ii) By a unique term $\Gamma \vdash \ulcorner t \urcorner : \sigma \longrightarrow \tau$ (same context, different type) such that $\Gamma \vdash_c t = \ulcorner t \urcorner c : \tau$.

So, contextual completeness, (i) above, means that the parameter $c : \sigma$ can be internalised in $\mathcal{L}$ by an extra variable $x : \sigma$ in every context. The parameter $c : \sigma$ can be 'instantiated' to actual constants of type σ, *cf.* the 'substitution' functors of §5.2, and thus we can think of a term $\Gamma \vdash_c t : \tau$ in $\mathcal{L}(c)$ as a function

$$(\vdash a : \sigma) \mapsto (\Gamma \vdash t[a/c] : \tau)$$

where $t[a/c]$ denotes the term with occurrences of c replaced by a. Functional completeness, (ii) above, means that such function can be internalised in $\mathcal{L}$ by a term $\Gamma \vdash \ulcorner t \urcorner : \sigma \longrightarrow \tau$. Thus, terms of type $\sigma \longrightarrow \tau$ internalise terms of type τ with a parameter of type σ. The categorical expression of these completeness properties is given in terms of categories $\mathbb{B}[x : I]$ with an indeterminate, as in §5.2. Note that the formulation there makes sense for a category $\mathbb{B}$ with a terminal object 1, the universal property holding for other such categories $\mathbb{C}$ and terminal object preserving functors $F : \mathbb{B} \to \mathbb{C}$ with a morphism $a : F1 \to FI$. This version is used in the following definition:

6.1.1. Definition. Let $\mathbb{B}$ be a category with a terminal object. We call $\mathbb{B}$

(i) *contextually complete* if for every $I \in \mathbb{B}$, the functor $\eta : \mathbb{B} \to \mathbb{B}[c : I]$ has a left adjoint;

(ii) *functionally complete* if every such η has a right adjoint.

This definition gives a finer formulation of the structure required in $\mathbb{B}$ to interpret a simply typed theory: the terminal object 1 interprets the empty context and thus closed terms of type τ correspond to global elements $\mathbb{B}(1, \tau)$, identifying types with their interpretation in $\mathbb{B}$. Terms with a free variable $x : \sigma$ correspond to closed terms in $\mathbb{B}[x : \sigma]$. To interpret these terms in $\mathbb{B}$, we require contextual completeness. Note that the left adjoint $L \dashv \eta_\sigma : \mathbb{B} \to \mathbb{B}[x : \sigma]$ determines a comonad on $\mathbb{B}$, which can be understood type-theoretically as performing 'context comprehension', as in [Jac92]:

$$\Gamma \mapsto \Gamma, x : \sigma$$

Note that contexts are inductively formed by context comprehension starting from the empty context. This is the reason to require finite products in $\mathbb{B}$ the above 'context comprehension' comonad is $\_ \times \sigma : \mathbb{B} \to \mathbb{B}$. A category with finite products is then contextually complete. Proposition 5.3.12 generalises this observation to cartesian objects in a 2-category. Thus, when the 2-category considered satisfies PCK, its cartesian objects are contextually complete.

By Proposition 5.2.1, $\mathbb{B}$ is functionally complete precisely when, for every σ, $\_ \times \sigma$ has a right adjoint, and thus $\mathbb{B}$ is cartesian closed. This is required for $\mathbb{B}$ to internalise the functions induced by terms with a parameter, as explained above.

The above analysis extends to polymorphic λ-calculi, $\lambda \to$ and $\lambda\omega$. Recall, from §2.1.3 that terms for these calculi are written as

$$\Theta \mid \Gamma \vdash t : \tau$$

where Θ is a context of type variables $X : \kappa$ in a kind κ, Γ is a context of term variables $x : \sigma$ in a type σ.

Let $\mathcal{P}$ be $\lambda \rightarrow$ or $\lambda\omega$. The presence of two levels of contexts, and thus two sorts of variables, leads us to consider contextual and functional completeness *for types* and *for kinds*. For types, we consider the polymorphic calculus $\mathcal{P}(c)$ with a new constant $c : \sigma$, where σ is a closed type *i.e.* $\vdash \sigma : \mathsf{Type}$. Its typing relation $\vdash_c$ extends $\vdash$ as before. Contextual completeness for types means that for each term $\Theta \mid \Gamma \vdash_c t : \tau$ in $\mathcal{P}(c)$, there is a unique term $\Theta \mid \Gamma, x : \sigma \vdash \llcorner t \lrcorner : \tau$ with

$$\Theta \mid \Gamma \vdash_c t = \ulcorner t \urcorner [c/x] : \tau.$$

Functional completeness for types means that for any such t there is a unique $\Theta \mid \Gamma \vdash \ulcorner t \urcorner : \sigma \longrightarrow \tau$ with

$$\Theta \mid \Gamma \vdash_c t = \ulcorner t \urcorner c : \tau.$$

To describe contextual and functional completeness for kinds, we consider a polymorphic calculus $\mathcal{P}(C, c)$ with a new constant $C : \kappa$, for a kind κ, and a new term constant $c : \sigma[X := C]$, for a type $X : \kappa \vdash \sigma : \Omega$ in $\mathcal{P}$. Its typing relation $\vdash_{C,c}$ extends $\vdash$ in the obvious way. Although one might expect completeness properties for kinds to be expressible in terms of the parameter C alone, the fact that types may involve occurrences of C

prompts the consideration of a parameter c. If we consider a calculus $\mathcal{P}(C)$ with only a a new constant C, terms $\Theta \mid \Gamma \vdash_C t : \tau$ in $\mathcal{P}(C)$ are such that the types of the term variables declared in Γ may involve occurrences of C, *e.g.* $x : C \in \Gamma$. Using finite products for types, we can assume there is only one such type in Γ, namely $X : \kappa \vdash \sigma : \Omega$. Then, to internalise occurrences of C in t we must also internalise those variables whose types involve C. This is the role of the parameter $c : \sigma[X := C]$.

So, completeness properties for kinds express the representability of terms $\Theta \mid \Gamma \vdash_{C,c} t : \tau$ of $\mathcal{P}(C, c)$ in $\mathcal{P}$, where the declarations $x_i : \tau_i$ of term variables in Γ are such that $\Theta \vdash \tau_i : \kappa_i$ in $\mathcal{P}$. That is the types occurring in Γ do not involve occurrences of C; the only type depending on C is σ. This is not relevant for contextual completeness, but is essential for the formulation of functional completeness.

Contextual completeness for kinds means that for each term $\Theta \mid \Gamma \vdash_{C,c} t : \tau$ in $\mathcal{P}(C, c)$ as above, there are unique $\Theta, X : \kappa \vdash \llcorner\tau\lrcorner : \Omega$ and $\Theta, X : \kappa \mid \Gamma, x : \sigma \vdash \llcorner t\lrcorner : \llcorner\tau\lrcorner$ such that $\Theta \vdash_{C,c} \tau = \llcorner\tau\lrcorner[X := C] : \Omega$ and $\Theta \mid \Gamma \vdash_{C,c} t = \llcorner t\lrcorner[X := C][x := c] : \tau$.

Functional completeness for kinds means that for each term $\Theta \mid \Gamma \vdash_{C,c} t : \tau$ there are unique $\Theta \vdash \ulcorner\tau\urcorner : A \longrightarrow \Omega$ and $\Theta \mid \Gamma\ulcorner t\urcorner : \Pi X : \kappa.(\sigma \longrightarrow \ulcorner\tau\urcorner X)$ with

$$\Theta \vdash_{C,c} \tau = \ulcorner\tau\urcorner C : \Omega$$

and

$$\Theta \mid \Gamma \vdash_{C,c} t = \ulcorner t\urcorner C c : \tau$$

Of course, functional completeness for kinds only makes sense for $\lambda\omega$.

Categorically, contextual and functional completeness for types are expressed in terms of fibrations with an indeterminate in $\mathcal{F}ib(\mathbb{B})$, while completeness properties for kinds involves fibrations with an indeterminate in $\mathcal{F}ib$. These are considered in §6.2 and §6.3 respectively.

6.2 Indeterminates for fibrations over a given base

We consider fibrations with an indeterminate element in $\mathcal{F}ib(\mathbb{B})$. First, we examine global elements and 2-cells in $\mathcal{F}ib(\mathbb{B})$. A global element $s : 1_{\mathbb{B}} \to p$ of $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ in $\mathcal{F}ib(\mathbb{B})$ corresponds to a family of objects $\{s(I) \in \mathbb{E}_I\}_{I \in \mathbb{B}}$ stable under reindexing. This means that for each $u : I \to J$ in $\mathbb{B}, u^*(s(J)) \cong s(I)$. In case the base category has a terminal object 1, the global element s is determined by the object $s(1)$ in the fibre over 1. Given such an object X in the fibre over 1, we write $s_X : 1_{\mathbb{B}} \to p$ for the corresponding global element, given by $I \mapsto !_I^*(X)$.

Let $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ be a fibration, with a terminal object 1 and a global element s. A 2-cell $\alpha : 1 \Rightarrow s$ is a family of vertical morphisms $\{\alpha_I : 1(I) \to s(I)$ in $\mathbb{E}_I\}_{I \in \mathbb{B}}$ stable under reindexing: for each $u : I \to J$ in $\mathbb{B}$ the following diagram is a pullback:

$$\begin{tikzcd} 1(I) \arrow[r, "1(u)"] \arrow[d, "\alpha_I"'] & 1(J) \arrow[d, "\alpha_J"] \\ s(I) \arrow[r, "s(u)"'] & s(J) \end{tikzcd}$$

When the base category $\mathbb{B}$ has a terminal object 1, the 2-cell $\alpha : 1 \Rightarrow s$ is determined by α_1.

Type-theoretically, when p is a $\lambda\to$- or $\lambda\omega$-fibration, a global element s corresponds to a closed type $\vdash s(1) : \Omega$, and a global 2-cell $\alpha : 1 \Rightarrow s$ corresponds to a closed term $\vdash \alpha_1 : s(1)$.

6.2.1. Remark. The equivalence $\mathcal{F}ib(\mathbb{B})(1_{\mathbb{B},p}) \simeq \mathbb{E}_1$, when $\mathbb{B}$ has a terminal object, as outlined above is an instance of Benabou's 'fibred Yoneda lemma'. See [Jac91a, Lemma 1.1.9]

A fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ is a cartesian object in $\mathcal{F}ib(\mathbb{B})$ when it has fibred finite products, *cf.* Definition 1.4.1. For a given global element $s : 1_{\mathbb{B}} \to p$, we write $p[c : s]$ for the fibration with an *indeterminate* 2-cell $c : 1 \Rightarrow s$, equipped with a fibred functor $\eta : p \to p[c : s]$ which preserves fibred finite products and a 2-cell $c : \eta 1 \Rightarrow \eta s$, universal among such, as specified in Definition 5.3.8. We can now formulate contextual and functional completeness for types as follows:

6.2.2. Definition. Let $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ be a fibration, with fibred finite products and $\mathbb{B}$

with a terminal object. $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ is

(i) *contextually complete for types* if for every global element $s_X : 1_{\mathbb{B}} \to p$, $\eta : p \to p[c : s_X]$ has a $\mathbb{B}$-fibred left adjoint, and

(ii) *functionally complete for types* if every such η has a $\mathbb{B}$-fibred right adjoint.

By Propositions 5.3.12 and 5.4.3 and Corollary 5.4.5, every fibration with fibred finite products is contextually complete for types. A concrete description of $p[c : s] : \mathbb{E}_{\_\times s} \to \mathbb{B}$ for a global element $s : 1_{\mathbb{B}} \to p$ induced by $X \in \mathbb{E}_1$ is the following: $\mathbb{E}_{\_\times s}$ has the same objects as $\mathbb{E}$, while its hom-sets are $(\mathbb{E}_{\_\times s})_I(Y, Z) = \mathbb{E}_I(Y \times_I !_I^*(X), Z)$. This agrees with the expected type-theoretic interpretation: 'contexts' in $p[c : s]$ have an extra variable of type X, 'weakened' to the appropriate kind context I.

We apply the above construction of $p[c : s]$ as a Kleisli fibration to show contextual and functional completeness for types $\lambda\to$- and $\lambda\omega$-fibrations. These are organised into 2-categories $\lambda\to$-$\mathcal{F}ib(\mathbb{B})$ and $\lambda\omega\to$-$\mathcal{F}ib(\mathbb{B})$ respectively, with structure preserving functors. We first show that Kleisli fibrations for fibred comonads inherit generic objects, as follows:

6.2.3. PROPOSITION. *Let* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *have a (strong) generic object* T *and let* $\langle G : p \to p, \epsilon, \delta\rangle$ *be a* $\mathbb{B}$*-fibred comonad. Then, the Kleisli fibration* $p_G : \mathbb{E}_G \to \mathbb{B}$ *has a (strong) generic object and* $U_G : p \to p_G$ *(the right adjoint of the resolution) preserves generic objects.*

Proof. It is routine to verify that T is also a (strong) generic object for p_G: for $X \in |\mathbb{E}|$, let $\chi_X : X \to T$ be a p-cartesian morphism in $\mathbb{E}$. Then $\chi_X \circ \epsilon_X : X \to T$ is p_G-cartesian in $\mathbb{E}_G$. Preservation is immediate. $\square$

The following lemma shows that $p[c : s]$ inherits $Cons_{\_}$-products from p.

6.2.4. LEMMA. *Consider a $\mathbb{B}$-fibred comonad $G : p \to p$ on* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$. *If p has $Cons_{\mathbb{B}}$-products then so does the Kleisli fibration* $\begin{smallmatrix}\mathbb{E}_G\\ \downarrow p_G\\ \mathbb{B}\end{smallmatrix}$ *and the fibred right adjoint $U_G : p \to p_G$ preserves them.*

Proof. The products $\overline{\Pi}$ for p_G are obtained from those of p, Π. Given I, J objects of $\mathbb{B}$, let $\pi : I \times J \to J$ be the projection. For any two objects $X \in |\mathbb{E}_J|$ and $Y \in |\mathbb{E}_{I\times J}|$, let $\overline{\Pi}_J(Y) \triangleq \Pi_J(Y)$ we have

$$
\begin{array}{rcl}
(\mathbb{E}_G)_{I\times J}(\pi^*(X), Y) & \cong & \mathbb{E}_{I\times J}(G(\pi^*(X)), Y) \\
& \cong & \mathbb{E}_{I\times J}(\pi^*(GX), Y) \\
& \cong & \mathbb{E}_J(GX, \Pi_J(Y)) \\
& \cong & (\mathbb{E}_G)_J(X, \overline{\Pi}_J(Y))
\end{array}
$$

The Beck-Chevalley condition for $\overline{\Pi}$ follows from that of Π. Preservation by $U_G : p \to p_G$ is immediate. $\square$

Now we can show

6.2.5. Proposition. *Let* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *be a* $\lambda \to$*-/*$\lambda\omega$*-fibration and let* $s : 1_{\mathbb{B}} \to p$ *be a global element. Then* $\eta : p \to p[c : s]$ *with* $c : \eta 1 \Rightarrow \eta s$ *as given in Proposition 5.3.12 exhibit* $p[c : s]$ *as the fibration with an indeterminate 2-cell* $c : 1 \Rightarrow s$ *in* $\lambda \to$*-/*$\lambda\omega$*-*$\mathcal{F}ib(\mathbb{B})$*.*

Proof. We must show that $p /\!\!/ s$ is a $\lambda \to$-/$\lambda\omega$-fibration if p is, and η preserves the relevant structure. Proposition 6.2.3 accounts for generic objects. Using the fibrewise presentation of $p[c : s]$, *cf.* Remarks 5.4.4, we conclude that its fibres are cartesian closed by [LS86, Part I, Proposition 7.1]; the structure is as given in $\mathbb{E}$. The reindexing functors for $p[c : s]$, which are those of p, preserve such structure. $Cons_{\mathbb{B}}$-products are similarly transferred, and preserved by η, by Lemma 6.2.4. It is easy to verify that the unique morphism induced into any other $\lambda \to$-/$\lambda\omega$-fibration given with the appropriate morphism into it and a global 2-cell, is a morphism in $\lambda \to$-/$\lambda\omega$-$\mathcal{F}ib(\mathbb{B})$.

6.2.6. Corollary. *Every* $\lambda \to$*-/*$\lambda\omega$*-fibration* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *is functionally complete for types.*

Proof. Given a global element $s : 1_{\mathbb{B}} \to p$, we must provide a $\mathbb{B}$-fibred right adjoint $R_s : p[c : s] \to p$ to η. It is given, fibrewise, as follows:

- on objects: $X \in |\mathbb{E}_I| \mapsto s(I) \Rightarrow X$
- on morphisms: given $f : X \times s(I) \to Y$ in $\mathbb{E}_I$, we have

$$s(I) \Rightarrow X \times s(I) \overset{\langle \mathrm{ev}, \pi' \rangle}{\to} X \times s(I) \overset{f}{\to} Y$$

and we get the desired morphism transposing the above one across the ex-ponential adjunction, $\Lambda(f \circ \langle \mathrm{ev}, \pi' \rangle) : s(I) \Rightarrow X \to s(I) \Rightarrow Y$.

$\square$

As expected, functional completeness involves λ-abstraction of term variables. This is reflected in the above proof in the definition of the morphism part of R_s

6.2.7. THE SIMPLE FIBRATION. For $\mathbb{B}$ a category with finite products, the assignment $I \mapsto \mathbb{B}[x : I]$ from objects of $\mathbb{B}$ to categories extends to a pseudo-functor $\mathbb{B}[\_ : \_] : \mathbb{B}^{op} \to \mathcal{C}at$ as follows: given a morphism $f : I \to J$ in $\mathbb{B}$, the reindexing functor $u^* : \mathbb{B}[y : J] \to \mathbb{B}[x : I]$ is determined by the functor $\eta_I : \mathbb{B} \to \mathbb{B}[X : I]$ and the morphism $\eta_I u \circ x : 1 \to \eta_I J$. Applying the Grothendieck construction to $\mathbb{B}[\_ : \_]$, we obtain the so-called *simple fibration* $\begin{smallmatrix} s(\mathbb{B}) \\ \downarrow \\ \mathbb{B} \end{smallmatrix}$; see [HJ93]. This fibration is a particular instance of a more general construction, presented in [Jac91a, §1.2.7], which we describe next. Given $\begin{smallmatrix} \mathbb{E} \\ \downarrow p \\ \mathbb{B} \end{smallmatrix}$ with fibred finite products, the fibration $s_p : s(\mathbb{E}) \to \mathbb{E}$, called *simple* of p, is defined as follows:

- The category $s(\mathbb{E})$ has

  Objects $|s(\mathbb{E})| = \{(X, X') \mid X, X' \in |\mathbb{E}|, pX = pX'\}$.

  Morphisms

  $$s(\mathbb{E})((X, X'), (Y, Y')) = \{(f, f') \mid f : X \to Y, f' : X \times X' \to Y', pf = pf\}$$

Composition is given by $(f, f') \circ (g, g') = (f \circ g, f' \circ \langle g \circ \pi, g' \rangle)$ and identity by $(1, \pi')$

- s_p acts on objects and morphisms as $(X, X') \mapsto X$. A morphism $(f, f') : (X, X') \to (Y, Y')$ is s_p-cartesian iff there is a vertical isomorphism $v : X \times X' \to X \times (pf')^*(Y')$ such that $f' = \overline{pf'}(Y') \circ \pi'_{X,(p,f')^*(Y')} \circ v$.

We show how the fibration s_p arises as the Kleisli fibration of a suitable fibred comonad in $\mathcal{F}ib(\mathbb{E})$. Consider the following pullback diagram

$$\begin{tikzcd}
\mathbf{E} \times_{p,p} \mathbf{E} \arrow[r, "\pi'_{p,p}"] \arrow[d, "\pi_{p,p}"'] \arrow[dr, "p \times p"] & \mathbf{E} \arrow[d, "p"] \\
\mathbf{E} \arrow[r, "p"'] & \mathbf{B}
\end{tikzcd}$$

This is both a change-of-base diagram, of p along p, and a binary product diagram in $\mathcal{F}ib(\mathbb{B})$ of p with itself. Recall that there is a fibred product functor $\times : p \times p \to$p. We define the following $\mathbb{E}$-fibred comonad $\langle G : \pi_{p,p} \to \pi_{p,p}, \epsilon, \delta \rangle$:

- $G = \langle \pi_{p,p}, \times \rangle$ (pairing in $\mathcal{F}ib(\mathbb{B})$). Concretely, $G(X, X') = (X, X \times X')$ and similarly for morphisms. $\pi_{p,p} \circ G = \pi_{p,p}$ and G preserves cartesian morphisms.

- $\epsilon : G \dot{\to} 1_{\pi_{p,p}}$ given by $\epsilon_{(X,X')} = (1_X, \pi'_{X,X}) : (X, X \times X') \to (X, X')$

- $\delta : G \dot{\rightarrow} G^2$ given by

$$\delta_{(X,X')} = 1_X, \langle \pi_{X,X'}, 1_{X \times X'} \rangle) : (X, X \times X') \rightarrow (X, (X \times X') \times X')$$

It is easy to verify that the above data yields an $\mathbb{E}$-fibred comonad and $s_p \cong p_G$. This is implicit in the description of the fibres $s(\mathbb{E})_X$ as polynomial categories $\mathbb{E}_{pX}[x : X]$ given in *ibid.*, since such categories correspond to $(\mathbb{E}_{pX})_{X \times \_}$ as we have already seen. Note also that the simple fibration $\begin{smallmatrix} s(\mathbb{B}) \\ \downarrow \\ \mathbb{B} \end{smallmatrix}$ is simple of $\begin{smallmatrix} \mathbb{B} \\ \downarrow \\ 1 \end{smallmatrix}$, the trivial fibration of $\mathbb{B}$ over 1.

There is a full and faithful fibred 1-cell $(H, 1) : p \rightarrow s_p$, as given in [Jac91a, §1.2.7].

It is worth spelling out the universal property of simple of p. Let $\mathcal{FPF}ib(\mathbb{B})$ be the 2-category of fibrations with fibred finite products over $\mathbb{B}$, fibred functors which preserve such products and the usual fibred 2-cells. Given $\begin{smallmatrix} \mathbb{E} \\ \downarrow p \\ \mathbb{B} \end{smallmatrix} \in \mathcal{FPF}ib(\mathbb{B})$, consider the two fibred functors $\Delta_p, 1_p : p \rightarrow p \times p$, where Δ_p is the diagonal functor and 1_p is given by $X \mapsto (X, 1(pX))$ (recall $1(pX)$ is the terminal object in the fibre $\mathbb{E}_{pX}$); *cf.* Definition 1.4.1. Then $\eta_p (= U_G) : p \times p \rightarrow s_p$ together with $\alpha : 1_p \Rightarrow \Delta$, given by $\alpha_X = (1_X, \pi_{X,1(pX)}) : (X, 1(pX)) \rightarrow (X, X)$ in $(\mathbb{E}_G)_{pX}$ have the following universal property in $\mathcal{FPF}ib(\mathbb{B})$: for any $\begin{smallmatrix} \mathbb{D} \\ \downarrow q \\ \mathbb{B} \end{smallmatrix}$ and any $H : p \times p \rightarrow q$ together with a 2-cell $\gamma : H1_p \Rightarrow H\Delta_p$, there is a unique (up-to-isomorphism) fibred functor $\overline{(H, \gamma)} : s_p \rightarrow q$ such that $\overline{(H, \gamma)}\eta_p \cong H$ and $\overline{(H, \gamma)}\eta_p\alpha = \gamma$ (modulo the given isomorphism between the functors), and similarly for 2-cells $\sigma : H \Rightarrow H'$ between two such functors, which determine a correspondent

$\overline{\sigma} : \overline{(H, \gamma)} \Rightarrow \overline{(H', \gamma')} : p \times p \to q$.

These simple fibrations play a central role in the semantics of type theories, as shown in [Jac91a].

6.2.8. REMARK. Using Corollary 3.3.11 we can infer that, if p is a $\lambda\omega$-fibration so is s_p and the 1-cell $(H, 1) : p \to s_p$ above preserves the relevant structure, extending [Jac91a, Theorem 3.3.3], which proves a similar result for $\lambda \to$- and $\lambda 2$-fibrations. As mentioned in [Jac91a, §3.3], such full and faithful structure preserving embedding of p into s_p for $\lambda 2$-fibrations constitutes the first step in Pitts' internalisation of a $\lambda 2$-fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ in the topos $\mathcal{S}et^{\mathbb{E}^{op}}$, obtaining a completeness results for topos-theoretic models of $\lambda 2$, [Pit87]. This is meant to proof that polymorphism is 'set-theoretic', provided we replace $\mathcal{S}et$ for the topos $\mathcal{S}et^{\mathbb{E}^{op}}$. The abovementioned fact about $\lambda\omega$-fibrations therefore allow us to extend Pitts' result to topos-theoretic models of $\lambda\omega$.

6.3 Indeterminates for fibrations in $\mathcal{F}ib$

We consider fibrations with an indeterminate element in $\mathcal{F}ib$. These will be used to show functional completeness for kinds for $\lambda\omega$-fibrations below, and to give semantics to ML-module features in §6.3.1. We first examine global elements and global 2-cells in $\mathcal{F}ib$.

The 2-category $\mathcal{F}ib$ has finite products. The fibration $\begin{smallmatrix}\mathbf{1}\\ \downarrow\\ \mathbf{1}\end{smallmatrix}$ written as $\mathbf{1}$, is terminal in $\mathcal{F}ib$, and the product of $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ and $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{A}\end{smallmatrix}$ is $\begin{smallmatrix}\mathbb{E}\times\mathbb{D}\\ \downarrow p\times q\\ \mathbb{B}\times\mathbb{A}\end{smallmatrix}$.

A global element $(\tilde{X}, X) : \mathbf{1} \to p$ of $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ in $\mathcal{F}ib$ amounts to an object $\tilde{X} \in \mathbb{E}$ above $X \in \mathbb{B}$. We thus write $X : 1 \to p$ for the global element (X, pX). In particular, the terminal object $1 \in \mathbb{E}$ forms such a global element $1 : 1 \to p$, when it exists. Type-theoretically, a global element corresponds to a type X with a free variable of kind pX.

A 2-cell $(f, g) : 1 \Rightarrow X$ between $1, X : \mathbf{1} \to p$ consists of a morphism $f : 1 \to X$ in the total category of p over $g : 1 \to pX$. It can therefore be identified with a morphism $u : 1 \to pX$ in the base category together with one $\hat{f} : 1 \to u^*(X)$ in fibre over 1. Therefore, such a 2-cell is written as $\langle u, \hat{f}\rangle : X$. Type-theoretically, u is a constant of kind pX and $\hat{f}$ a term constant of type $u^*(X)$.

By Definition 3.3.4, a fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ has finite products, *i.e.* it is a cartesian object in $\mathcal{F}ib$, if both $\mathbb{E}$ and $\mathbb{B}$ have finite products and p strictly preserves them. Equivalently, by Corollary 3.3.6, $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ has finite products if the base category $\mathbb{B}$ has finite products and p has fibred finite products, *i.e.* it is a cartesian object in $\mathcal{F}ib(\mathbb{B})$. A morphism $(\tilde{H}, H) : p \to q$ in $\mathcal{F}ib$ preserves finite products if both $\tilde{H}$ and H preserve them in the ordinary sense.

For a fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ with finite products and a global element $X \in |\mathbb{E}|$, we can describe the fibration with an indeterminate $p[\langle C, c\rangle : X]$ in $\mathcal{F}ib$, instantiating Definition 5.3.8. This is used to express contextual and functional completeness for kinds, as follows:

6.3.1. Definition. Let $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ be a fibration with finite products. p is

(i) *contextually complete for kinds* if, for every $X \in |\mathbb{E}|, \eta : p \to p[\langle C, c\rangle : X]$ has a left adjoint in $\mathcal{F}ib$, and

(ii) *functionally complete for kinds* if every such η has a right adjoint in $\mathcal{F}ib$.

This categorical expression of completeness for kinds properly reflects the type-theoretic version for polymorphic λ-calculi in §6.1.

By Proposition 5.3.12, Theorem 5.4.11 and Corollary 5.4.13, a fibration with finite products is contextually complete. We want to extend this result for $\lambda\to$- and $\lambda\omega$-fibrations and show that $\lambda\omega$-fibrations are functionally complete for kinds. We must first show that $p[\langle C, c\rangle : X]$ is a $\lambda\to$-/$\lambda\omega$-fibration when p is. To do so we need the following auxiliary results:

6.3.2. Proposition. *Given* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$, *with a (strong) generic object G (ouer Ω) and an adjunction $F \dashv U : \mathbb{B} \to \mathbb{A}$ (via η, ϵ), let $(\overline{U}, U)p \to F^*(p)$ be the fibred right adjoint to $(p^*(F), F)$ induced by change-of-base, as in Lemma 3.2.1. Then, $F^*(p)$ has a (strong) generic object and $(\overline{U}, U)$ preserves generic objects.*

Proof. We must simply verify that $\overline{U}G(= \langle \epsilon^*_\Omega(G), G\Omega\rangle)$ over $G\Omega$ is a (strong) generic object for $F^*(p)$. Let X be an object of $F^*(\mathbb{E})$. We obtain a cartesian morphism $\chi_X : X \to \overline{U}G$ as the adjoint transpose of a cartesian morphism $\chi_{p^*(F)X} : p^*(F)X \to G$, *cf.* Remark 1.2.13. The hom-set isomorphism

$\mathbb{A}(F^*(p)(X), U(\Omega)) \cong \mathbb{B}(F(F^*(p)(X)), \Omega)$ implies that if G is strong, so is $\overline{U}G$. □

6.3.3. COROLLARY. *Let* $\begin{smallmatrix}\mathbb{D}\\ \downarrow q\\ \mathbb{B}\end{smallmatrix}$ *be a fibration with a fibred terminal object* $1 : 1_{\mathbb{B}} \to q$. *Given another fibration* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *, if* p *has a (strong) generic object, so does* $q^*(p)$ *and the fibred 1-cell* $(\overline{1}, 1) : p \to q^*(p)$ *preserves generic objects.*

6.3.4. PROPOSITION. *Consider* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *where* $\mathbb{B}$ *has finite products. Let* I *be an object of* $\mathbb{B}$. Consider the comonad $\_ \times I : \mathbb{B} \to \mathbb{B}$ *and its Kleisli resolution* $F \dashv U : \mathbb{B} \to \mathbb{B}_{\_\times I}$. *If* p *has* $Cons_{\mathbb{B}}$ *-products then* $F^*(p)$ *has* $Cons_{\mathbb{B}_{\_\times I}}$*-products and the fibred 1-cell* $(\overline{U}, U) : p \to F^*(p)$ *preserves* $Cons_{\_}$*-products.*

Proof. Given K, J objects of $\mathbb{B}$ the image of the projection $\overline{\pi} : K \times J \to J$ in $\mathbb{B}_{\_\times I}$ along F is isomorphic to the projection $\pi : (I \times K) \times J \to I \times J$, hence

$$\overline{\pi}^{*F^*(p)} \cong \pi^{*p} \dashv \Pi_J^{I \times K}$$

A simple calculation shows that such right adjoints satisfy the Beck-Chevalley condition for $F^*(p)$ because they satisfy it for p. The Beck-Chevalley condition for p is used once more to show that $(\overline{U}, U) : p \to F^*(p)$ preserves $Cons_{\_}$-products. □

Let $\lambda \to$-$\mathcal{F}ib$ and $\lambda\omega$-$\mathcal{F}ib$ be the 2-categories of $\lambda \to$- and $\lambda\omega$-fibrations respectively, with structure preserving fibred 1-cells. Given the description of $p[\langle C, c\rangle : X]$ as a Kleisli fibration for a comonad in $\mathcal{F}ib$, we have:

6.3.5. Corollary. *Given a fibration* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ *and a global element* $X \in |\mathbb{E}|$, *if* $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ is a $\lambda \rightarrow$-/$\lambda\omega$-*fibration, so is* $p[\langle C, c\rangle : X]$ *and* $\eta : p \to p[\langle C, c\rangle : X]$ *preserves the relevant structure. Furthermore, the universal property of* η *holds in* $\lambda \rightarrow$-/$\lambda\omega$-$\mathcal{F}ib$.

Proof. We first show that $F^*(p)$ has the relevant structure. The fibred-ccc structure of $p[\langle C, c\rangle : X]$ is obtained from that of p because $F^* : \mathcal{F}ib(\mathbb{B}) \to \mathcal{F}ib(\mathbb{B}_{\_\times I})$ preserves finite products and groupoid fibrations and hence it preserves the relevant fibred adjunctions. The presence of a generic object follows from Proposition 6.3.2. $Cons_{\mathbb{B}_{\_\times I}}$-products are obtained by Proposition 6.3.4. Knowing that $F^*(p)$ has the relevant structure, we prove that $p[\langle \alpha, x\rangle : X]$ has it as well, using Proposition 6.2.3 and Lemma 6.2.4. The rest of the corollary is proved with similar arguments. $\square$

A concrete description of $p[\langle C, c\rangle : X]$ goes as follows: its total category, written $\mathbb{E}/\!\!/(X)$d, has:

Objects: (I, Y) with $I \in \mathbb{B}$, $Y \in \mathbb{E}$ and $pY = I \times pX$.

Morphisms: $(u, f) : (I, Y) \to (J, Z)$ with $u : I \times pX \to J$ and $f : Y \times (\pi'_{I,pX})^*(X) \to Z$ over $\langle u, \pi'_{I,pX}\rangle : I \times pX \to J \times pX$.

$p[\langle C, c\rangle : X]\mathbb{E}/\!\!/(X) \to \mathbb{B}/\!\!/pX$ — where $\mathbb{B}/\!\!/pX$ denotes the Kleisli category of $\_ \times pX$ — takes objects and morphisms to their first components. $\eta =$

$(U', \overline{U}, U) : p \to p[\langle C, c\rangle : X]$ acts as follows: for $I \in |\mathbb{B}|$, $UI = I$, and for $Y \in |\mathbb{E}_I|$,$U'\overline{U}Y = (I, \pi^*_{I,pX}(Y))$.

Note that a morphism in $(u, f) : \eta(Y) \to (J, Z)$ in $\mathbb{E}/\!\!/(X)$ corresponds to a term $pY \mid \pi^*_{pY,pX}(Y) \vdash_{C,c} f : Z(u, c)$, with $C : pX$ and $c : X(C)$, where the 'context for type varibles' $\pi^*_{pY,pX}(Y)$ does not depend on C. This reflects precisely the restriction on contexts required in the formulation of functional completeness for kinds in §6.1

$\lambda \to$-/ and $\lambda\omega$-fibrations are contextually complete for kinds, in view of the presentation of the corresponding fibrations with indeterminates as Kleisli fibrations above. We now show that $\lambda\omega$-fibrations are also functionally complete for kinds.

6.3.6. PROPOSITION. *Every $\lambda\omega$-fibration is functionally complete for kinds.*

Proof. Let $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ be a $\lambda\omega$-fibration. We must construct a fibred right adjoint $(\tilde{R}, R)$ to $(U'\overline{U}, U) : p \to p[\langle C, c\rangle : X]$. The base right adjoint $R : \mathbb{B}_{\_\times pX} \to \mathbb{B}$ is given by $I \mapsto pX \Rightarrow I$. $\tilde{R} : \mathbb{E}/\!\!/(X) \to \mathbb{E}$ is obtained as the composite $\hat{R}\overline{R}R'$, applying Theorem 3.2.3 where:

- $R' : \mathbb{E}/\!\!/(X) \to F^*(\mathbb{E})$ is the right adjoint to $U' : F^*(\mathbb{E}) \to \mathbb{E}$, given by $(I, Y) \mapsto (I, (\pi'_{I,pX})^*(X) \Rightarrow_{I\times pX} Y$

- $\overline{R} : F^*(\mathbb{E}) \to U^*F^*(\mathbb{E})$ is given by Lemma 3.2.1:

$$(I, Y) \mapsto \langle \mathrm{ev}_{pX,I}, \pi'_{pX\Rightarrow I,pX}\rangle^*(Y)$$

- $\hat{R} : U^*F^*(\mathbb{E}) \to \mathbb{E}$ is the right adjoint to $\hat{\overline{U}} : \mathbb{E} \to U^*F^*(\mathbb{E})$ — whose action is $Y \mapsto \pi^*_{pY,pX}(Y)$ — given by $(I, Y) \mapsto \Pi_K(Y)$.

Hence $\tilde{R}$ has action

$$(I, Y) \mapsto \Pi_K((\pi'_{pX\Rightarrow I,pX})^*(X) \Rightarrow_{pX\Rightarrow I\times pX} \langle \mathrm{ev}_{pX,I}, \pi'_{pX\Rightarrow I,pX}\rangle^*(Y))$$

$\square$

Note that functional completeness of a $\lambda\omega$-fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ is due to the fact that both $\mathbb{B}$ and $\mathbb{E}$ are cartesian closed, the latter by Corollary 3.3.11.

6.3.1 A semantics for ML-style modules using polynomial fibrations

In [FP92], a topos theoretic semantics for Pure ML is presented. The approach advocates *synthetic domain theory* to interpret recursive functions and data types, and the theory of *classifying toposes* via the notion of *generic structure* to interpret signatures, structures and functors. Here we adapt the latter idea and show how fibrations with indeterminates can interpret ML-style signatures, structures and functors. For simplicity, we consider the purely functional fragment of SML without recursion, *i.e.* a simply typed λ-calculus with type variables as embodied in the notion of a $\lambda \to$-fibration; this is the minimal setting required to illustrate the above application.

Consider the following ML-signature:

```
signature Order =
  sig
```

```
      type t;
      val le:  t * t → bool
   end;
```

Thus, a signature declares types (`t` above) and values (`le` above) whose type may involve the declared types besides the 'pervasive' types (`bool` above). They may also contain structures, but we will consider them later.

Consider $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ a $\lambda \rightarrow$-fibration, which interprets a $\lambda \rightarrow$-calculus with the basic types of ML. Recall that the generic object T lies over the kind Ω of all types, *e.g.* $\mathrm{bool} : 1 \to \Omega$ names the (closed) type $B = \mathrm{bool}^*(T)$ corresponding to `bool` above. Similarly, $t : 1 \to \Omega$ names `t` above. The value `le` corresponds to a morphism $\mathrm{le} : 1 \to T \times T \Rightarrow !_\Omega^*(B)$, which we identify with its vertical factor $\mathrm{le} : 1 \to t^*(T \times T \Rightarrow !_\Omega^*(B))$. Therefore we interpret the above signature `Order` by adjoining $t : \Omega$ and $\mathrm{le} : t^*((T \times T \Rightarrow !_\Omega^*(B))$ to p:

$$[\![\mathtt{Order}]\!] = p[\langle t, \mathrm{le}\rangle : T \times T \Rightarrow !_\Omega^*(B)] : \mathbb{E}/\!\!/(T \times T \Rightarrow !_\Omega^*(B)) \to \mathbb{B}/\!\!/\Omega$$

Now consider the following structure which matches the signature `Order`:

```
structure IntOrder:  Order =
   struct
      type int;
      fun le(m,n) = (in =< n)
   end;
```

Thus a structure amounts to a choice of the components declared in the signature. Hence we can interpret the structure `IntOrder` as the morphism $\llbracket$`IntOrder`$\rrbracket$ of $\lambda\to$-fibrations in the diagram below, determined by the universal property of $\llbracket$`Order`$\rrbracket$.

$$\begin{tikzcd}
\llbracket Order \rrbracket \arrow[r, "\llbracket IntOrder \rrbracket"] & p \\
p \arrow[u, "\eta"] \arrow[ur, "1 \text{ with } \langle \mathrm{int}, \mathrm{le} \rangle : 1 \to T \times T \Rightarrow !^*_{\Omega}(B)"']
\end{tikzcd}$$

where the le being substituted is that defined in the structure `IntOrder`.

With respect to ML-functors, consider

```
functor Dual(structure O1:Order):Order =
   struct
      type t = O1.t;
      fun le(x,y) = O1.le(y,x)
   end;
```

The above ML-functor takes a structure matching `Order` as argument and produces another such structure, namely one with the same type but with the dual ordering relation. Note that the mapping

$$\mathtt{O1.t} \mapsto \mathtt{dual(O1).t} \qquad \mathtt{O1.le} \mapsto \mathtt{dual(O1).le}$$

determines a 2-cell

$$\langle 1_{\Omega}, s \Rightarrow B \rangle : \langle \Omega, t^*(T \times T) \Rightarrow B \rangle \Rightarrow \langle \Omega, t^*(T \times T) \Rightarrow B \rangle$$

where $s : t^*(T \times T) \to t^*(T \times T)$ is the canonical 'swap' (or symmetry) isomorphism $s = t^*\langle \pi', \pi \rangle$. By the universal property of $[\![\texttt{Order}]\!]$, such a 2-cell induces a $\lambda \to$-morphism $[\![\texttt{dual}]\!] : [\![\texttt{Order}]\!] \to [\![\texttt{Order}]\!]$, which is the interpretation of the ML-functor dual. The action of `dual` on structures is given by precomposition with $[\![\texttt{dual}]\!]$.

Signatures containing structures are interpreted by iterating the process of forming fibrations with indeterminates. Consider for instance the signature

```
signature Sig =
  sig
    type t;
    structure str:  Sig'
    val v: ... str.t' ...
  end;
```

The interpretation of `Sig` is built upon the interpretation of `Sig'` as follows: the fibration $[\![\texttt{Sig'}]\!]$ has a generic structure matching `Sig'`. Thus we can interpret `Sig` over this fibration by $[\![\texttt{Sig}]\!] = [\![\texttt{Sig'}]\!][\langle t, v\rangle : \ldots\ (t')^*(T)\ \ldots\ ]$.

Regrettably, the 2-category $\lambda \to$-$\mathcal{F}ib$ does not seem to have enough structure to interpret features like sharing.

Chapter 7

Conclusions and further work

The aim of this thesis was to give a category-theoretic account of certain logical phenomena, *i.e.* logical predicates, induction and indeterminates. These topics are important in the semantics of type theories and programming languages and therefore a proper abstract account of them is convenient, and even necessary, for their application as well as their generalisation to other systems.

Our approach has been to investigate the above topics using the correspondence category of predicates $\equiv$ fibred category. This identification is at the right level of abstraction to unveil some of the abstract results which underlie the above logical phenomena.

Thus, we have shown how certain properties of a fibred category, *i.e.* cartesian closure, can be interpreted logically, via the internal language of the fibration, to obtain logical predicates for simply typed λ-calculus, *cf.* Corollary 3.3.11 and Definition 4.2.2. The main property of logical predicates, namely the Basic Lemma, has a clear expression in this context as the soundness of typing for interpretations of simply typed λ-calculus in a cartesian closed category, *cf.* Corollary 4.2.4.

An analogous argument allows us to explain the induction principle for inductive data types categorically, in terms of preservation of initial algebras for an endofunctor on a distributive category, *cf.* Definition 4.5.13. Here again, we recover the logical view of induction via the internal language of the fibration. This shows the adequacy of our treatment.

An abstract account of a concept should stand on its own: if we have captured the essential features of a problem, we should be able to prove the right results with the right hypotheses. In particular, an abstract account cannot be justified solely in terms of examples, although it is important to recover these as particular instances. A mere rephrasing of concrete examples using categorical language is quite often misleading if irrelevant features are taken into account. A significative example is the identification predicate $\equiv$ subobject, justified for higher-order logic and $\mathcal{S}et$-like settings, like a topos. However, when analysing logical predicates for simply typed λ-calculus, which are expressed in first-order logic, we should not avail ourselves of the comprehension principle implicit in the consideration of a predicate as a morphism in the category. In particular, we should not prove the cartesian closed property of a category of predicates using the domain of a subobject, which corresponds to the extent of a predicate, as in [MR91], if we want to achieve the right level of generality.

Thus, our account succeeds in proving the relevant results using suitable hypotheses, namely the logical connectives and quantifiers supported by the structure of a fibred category. We have examined a few illustrative examples in this setting. In particular, we have seen in §4.3.2 that certain desirable logics do not have all the structure necessary to make sense of logical predicates, as far as the individual connectives and quantifiers in the relevant formulas are concerned. However, these logics may be able to interpret the relevant formulas all the same, because of their capability of internalising the structure of a richer, external logic over the same category. In the case of $\omega\mathcal{C}po$, we have exploited the fact that exponentials in it are obtained from those in $\mathcal{S}et$, with a pointwise cpo structure, a standard property of categories of sets with structure. Of course, much remains to be done in the way of applications.

In the case of the induction principle, the categorical conditions on a 'logic' over a distributive category in Proposition 4.5.8 give the appropriate logical expressivity to make sense of induction, *cf.* Examples 4.5.14. Note that for this abstract notion of logic over a distributive category, the induction principle is a property we must impose on it. It cannot be taken for granted, although it holds when we restrict ourselves to internal logics, by the presence of comprehension *cf.* Corollary 4.5.15.

We would like to emphasise the importance of a 2-categorical perspective in studying these issues. One of the main results of the thesis, Theorem 3.2.3, which underlies the categorical account of logical predicates and induction outlined above, is a property of fibrations qua objects in the 2-category

$\mathcal{F}ib$. This is the framework in which we have developed our results about fibrations.

We have also dealt with indeterminate elements for fibrations with some structure, as relevant for the interpretation of polymorphic λ-calculi. We have thus been able to capture some aspects of parameterisation in these calculi, as embodied in the notions of contextual and functional completeness for kinds and types, *cf.* Definitions 6.3.1 and 6.2.2. To do so, we reformulated cartesian categories with indeterminates elements 2-categorically in §5.3. We proved in this general context Lambek's characterisation of these categories with indeterminates as Kleisli objects for certain comonads in Proposition 5.3.12. This result relies essentially on Street's presentation of Kleisli objects for comonads as oplax colimits, *cf.* Definition 5.3.3, and the observation that $\mathcal{C}at$, as well as $\mathcal{F}ib(\mathbb{B})$ and $\mathcal{F}ib$, satisfy a special property, called PCK in Definition 5.3.9, which asserts that finite products in these 2-categories preserve certain oplax colimits. To apply this result in $\mathcal{F}ib(\mathbb{B})$ and $\mathcal{F}ib$, we have shown the existence of Kleisli objects for comonads in these 2-categories. Theorem 5.4.11, which presents the construction of such Kleisli objects in $\mathcal{F}ib$, constitutes another of the main technical results of this thesis.

Both of these main results, Theorems 3.2.3 and 5.4.11, have been proved by 2-categorical diagram chasing, using the algebraic laws of fibred 2-cells in §3.1.1. These arguments rely thus on the fact that $\mathcal{F}ib$ is fibred as a 2-category over $\mathcal{C}at$: there is a cartesian-vertical factorisation of l-cells, given by pullbacks, as well as for 2-cells, as given in Lemma 3.1.2.(ii). Thus the abovementioned theorems hold in any such fibred 2-category. We regard this as the appropriate level of abstraction to study 2-dimensional aspects of categorical logic, as pointed out by Bénabou. For instance, fibred 2-categories are the common framework underlying our semantics of ML module features in §6.3.1 and the one in [FP92], on which our approach was based. The latter is set in the following fibred 2-category: the base is the 2-category of toposes and geometric morphisms and the fibre over a topos $\mathcal{E}$ is the 2-category of internal full subcategories in it. This is a sub-fibred-2-category of $\mathcal{F}ib$ over $\mathcal{C}at$. This set-up is not mentioned in *ibid.* though. In both cases, signatures, modules and functors are understood in terms of indeterminates for the objects of the corresponding fibred 2-category. In the second case, due to the presence of additional structure in the objects considered, objects with indeterminates are not given by a Kleisli construction as in the simpler setting we considered in §6.3.1. Of course, the presence of considerably additional structure in the framework of *ibid.* allows a fuller account of ML features than ours.

A glimpse at the proofs in §5.4.2 prompts a study of coherence problems for fibred bicategories. [Pow93] presents a vivid account of such coherence problems.

A natural continuation of the work in this thesis, is to give an account of logical predicates for general type systems, in the sense of [Jac91a], including polymorphic λ-calculi and the Calculus of Constructions. To do so, we need to consider fibrations in 2-categories such as $\mathcal{F}ib(\mathbb{B})$, $\mathcal{C}at(\mathbb{B})$ and $\mathcal{F}ib$. Fibrations in $\mathcal{F}ib(\mathbb{B})$ have been studied by Bénabou and applied to type theory in [Jac91a, Pav90]. Fibrations in the above 2-categories give the setting to study logic for type systems, *e.g.* $\lambda 2$, which are themselves understood as fibrations. For instance, Theorem 3.2.3 can be applied to adjunctions between fibrations in $\mathcal{F}ib(\mathbb{B})$ and gives as direct consequences some results about 'lifting' of products and sums for such fibrations proved in [Pav90, Prop. II.3.73]. Such results are relevant to the characterisation of logical predicates for type systems; the latter reflect categorical properties of the logics for a type system. Of course, the relevant technical machinery is more involved than that for the simple type systems we have considered in this thesis.

It also remains to investigate notions of logical predicates for calculi such as linear logic, where some connectives, like $\otimes$, do not have a universal property, unlike $\times$ for instance, and thus cannot be handled in terms of adjoints. [Amb92] has some relevant results in this direction.

Another important direction of research is the study of logics via fibrations for type systems with partial terms. A starting point is to consider fibrations over categories of partial maps. We have some preliminary results about such fibrations, e.g. suitable conditions to extend a fibration $\begin{smallmatrix}\mathbb{E}\\ \downarrow p\\ \mathbb{B}\end{smallmatrix}$ from $\mathbb{B}$ to a category of partial maps of $\mathbb{B}$ specified by a dominion [RR88]. A salient feature of these categories of partial maps is that their horn-sets are partially ordered, and so they are 2-categories. Thus, in a logic for such categories, the primary relation between 'terms' $t, t' : A \to B$ is not that of equality $t = t'$ as in algebraic theories, but the order relation $t \leq t'$. In the internal logic, the predicate $x : A \vdash tx \leq t'x$ is given bv the inserter of t and t', *i.e.* the universal 1-cell $e : A' \to A$ such that $te \leq t'e$; see [Kel89] for a precise definition. This seems the appropriate 2-dimensional analogue of the fact that the equality predicate $x : A \vdash tx = t'x$ is given by the equaliser of t and t'.

In the framework of category, as given in [Str73, Joh92], as the categorical formulation of the notion of family, which in the 1-dimensional case are usually presented as display maps [Jac90]. This 2-categorical aspect of fibrations is of advantage over indexed-categories. It should be of relevance to understanding dependent types in categories of partial maps.

The above considerations on logics over 2-categories, as arising from categories of partial maps for instance, is a starting point to study the categorical-logical aspects of reduction properties in type systems, considering the interpretation of term-rewriting in a 2-category as in [RS87].

We should investigate further applications of the construction of categories and fibrations with indeterminate elements. For instance, we know that functional completeness for a cartesian closed category $\mathbb{C}$, as expressed in §5.2, can be used to derive combinators for simply typed λ-calculus, by regarding the category $\mathbb{C}[x : I]$ with an I-indeterminate as a category enriched over $\mathbb{C}$ [Kel82]. It remains to study indeterminates for type theories with dependent types, as embodied in the notion of comprehension category of [Jac92].

Finally, we must also consider (co)induction principles for coinductive and recursive data types, involving exponentials, extending the approach in §4.5. This should shed light in the formulation of such principles. Of course, the incorporation of these concepts in logics for partial maps and dependent types would be quite challenging.

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